

## Classification of discrete integrable equations of Hirota type

V.E. Adler, A.I. Bobenko, Yu.B. Suris

One of possible definitions of integrability of discrete equations is based on the notion of multidimensional consistency. We apply this approach for classification of integrable 3-dimensional equations of Hirota, or discrete KP-type. It is proven, under rather general and natural assumptions, that the list of such equations is exhausted by Hirota equation itself and its three modifications.

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## Plan

- Multidimensional consistency
  - 3D consistency
  - 4D consistency of Hirota-Miwa equation
  - Hirota equation: consistent triple
  - Hirota equation: consistent quintuple
  - Another example: Desargues configuration
- **Classification theorem**
- Three-leg forms of Hirota-type equations
  - Three-leg forms as a mechanism of consistency
  - From consistent quintuple to a single equation
  - Classification of three-leg equations
  - From single equation to consistent quintuple

## Notations

- $x$  denotes a map  $\mathbb{Z}^d \rightarrow \mathbb{R}$
- the arguments are omitted:  $x = x(n_1, \dots, n_d)$
- the subscripts denote partial shifts:

$$x_i = T_i(x) = x(\dots, n_i + 1, \dots)$$

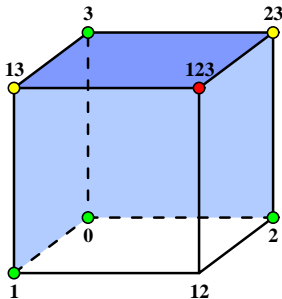
- all equations are assumed autonomous, that is their coefficients do not depend on  $n_1, \dots, n_d$

## 3D consistency

An equation of discrete KdV-type

$$f(x, x_i, x_j, x_{ij}) = 0$$

is called 3D-consistent, or consistent around a cube, if the value  $x_{123}$  as the function on initial data  $x, x_1, x_2, x_3$  does not depend on the order of computation.



Typical examples:

discrete KdV:  $(x - x_{ij})(x_i - x_j) = a^{(i)} - a^{(j)}$

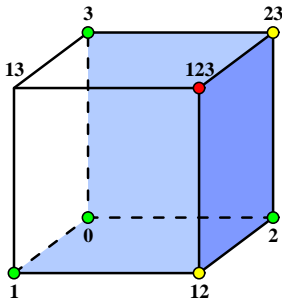
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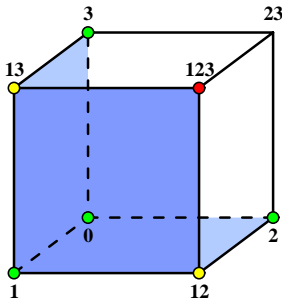
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## 4D consistency of Hirota-Miwa (or dBKP, or dNVN) equation

The following set of four equations

$$x_1x_{23} - x_2x_{13} + x_3x_{12} - xx_{123} = 0,$$

$$x_1x_{24} - x_2x_{14} + x_4x_{12} - xx_{124} = 0,$$

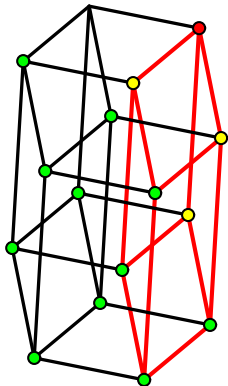
$$x_1x_{34} - x_2x_{14} + x_4x_{13} - xx_{134} = 0,$$

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is 4D-consistent, that is the value  $x_{1234}$  as the function on initial data  $x$ ,  $x_i$ ,  $x_{ij}$  does not depend on the order of computation.

Remarkably, these equations imply a similar equation on odd/even sublattices in  $\mathbb{Z}^4$ :

$$x_{14}x_{23} - x_{13}x_{24} + x_{12}x_{34} - xx_{1234} = 0.$$



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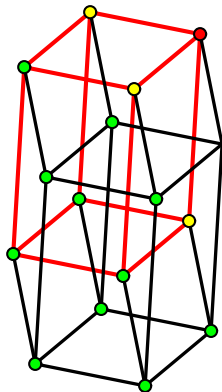
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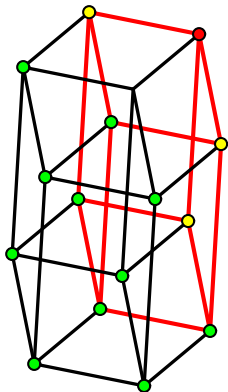
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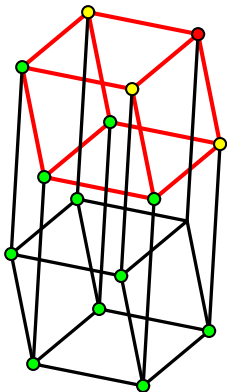
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Another example is the **double cross-ratio** equation

$$\frac{(x - x_{ij})(x_{jk} - x_{ki})}{(x_{ij} - x_{jk})(x_{ki} - x)} = \frac{(x_{ijk} - x_k)(x_i - x_j)}{(x_k - x_i)(x_j - x_{ijk})}.$$

Again, the value  $x_{1234}$  does not depend on the order of computation, although in this case no equation appears on odd/even sublattice.

Consistency property is a discrete version of the notion of higher symmetry for integrable equations. In contrast to 2D case, only few 3D integrable equations are known. Double cross-ratio and several other modifications are related to Hirota-Miwa equation via certain difference substitutions; another example is the discrete CKP equation.

However, the classification problem for this type of equations is very difficult and we address here to a bit more simple class of **Hirota-type** equations.

## Hirota (dKP) equation

dKP equation can be obtained from dBKP

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through a limiting process. Notice that the scaling

$$x(m, n, k) \rightarrow a^{mn} b^{nk} c^{mk} x(m, n, k)$$

brings dBKP to the form

$$bx_1x_{23} - cx_2x_{13} + ax_3x_{12} - abcxx_{123} = 0,$$

so that the last term is dropped under the limit  $a = b = c \rightarrow 0$ .

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But this changes the combinatorics of equation; two questions should be answered:

- 1) which set of equations is consistent?
- 2) how to define the consistency?

A hint for the **question 1)** comes from the consistent quadruple of dBKP equations. One obtains, by the same limiting process for all equations:

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However, this set of four equations is not independent: one equation follows from the other three. Moreover, the equation on odd/even sublattice also becomes a consequence of these three.

(Let us see how these  $\uparrow\downarrow$  can be proven. Later on we will see that this is not just a trick!)

$$x_1x_{23} - x_2x_{13} + x_3x_{12} = 0$$



$$\frac{x_{12}}{x_1x_2} - \frac{x_{13}}{x_1x_3} + \frac{x_{23}}{x_2x_3} = 0$$



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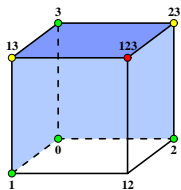
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As the answer on the **question 2)**, it is natural to introduce the notion of consistency in terms of three equations which remain independent. This is actually a logical step back to 3D-consistency situation.



**Definition of consistent triple.** Equations

$$\begin{aligned} x_{12} &= f(x_1, x_2, x_4, x_{14}, x_{24}), \\ x_{13} &= g(x_1, x_3, x_4, x_{14}, x_{34}), \\ x_{23} &= h(x_2, x_3, x_4, x_{24}, x_{34}) \end{aligned} \quad (1)$$

are called 4D-consistent if the equalities

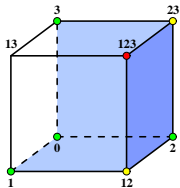
$$\begin{aligned} x_{123} &= f(g, h, x_{34}, T_4(g), T_4(h)) = g(f, h, x_{24}, T_4(f), T_4(h)) \\ &= h(f, g, x_{14}, T_4(f), T_4(g)) \end{aligned} \quad (2)$$

hold identically on the initial data

$$x_1, x_2, x_3, x_4, x_{14}, x_{24}, x_{34}, x_{44}, x_{144}, x_{244}, x_{344}.$$

**The role of 4-th coordinate is distinguished, but we will prove soon that the symmetry can be restored.**

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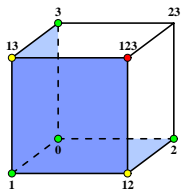
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## Remark: a continuous analog

There exist 4D-consistent triples of 3D dispersionless PDE of the form (now, subscripts denote derivatives)

$$\begin{aligned}u_{xy} &= f(u_x, u_y, u_t, u_{xt}, u_{yt}), \\u_{xz} &= g(u_x, u_z, u_t, u_{xt}, u_{zt}), \\u_{yz} &= h(u_y, u_z, u_t, u_{yt}, u_{zt}).\end{aligned}$$

This means that the cross-derivatives must coincide:

$$u_{xyz} = D_z(f) = D_y(g) = D_x(h)$$

where  $D_x, D_y, D_z$  are total derivatives in virtue of the system, e.g.

$$D_x(h) = \frac{\partial h}{\partial u_y} f + \frac{\partial h}{\partial u_z} g + \frac{\partial h}{\partial u_t} u_{xt} + \frac{\partial h}{\partial u_{yt}} D_t(f) + \frac{\partial h}{\partial u_{zt}} D_t(g).$$

The triples look quite similar to the discrete ones, for example the following system is consistent:

$$(b - a)u_t u_{xy} - bu_x u_{ty} + au_y u_{tx} = 0,$$

$$(a - c)u_t u_{zx} - au_z u_{tx} + cu_x u_{tz} = 0,$$

$$(c - b)u_t u_{yz} - cu_y u_{tz} + bu_z u_{ty} = 0.$$

Moreover, the equation

$$(a - b)u_z u_{xy} + (c - a)u_y u_{xz} + (b - c)u_x u_{yz} = 0$$

follows, so that all variables are on equal footing.

## From triple to quintuple

**Theorem 1.** If the triple (1) is consistent then some equations

$$k(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) = 0, \quad (3)$$

$$l(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = 0 \quad (4)$$

are fulfilled automatically.

**Proof.** Differentiating the consistency condition (2) and eliminating the derivatives of composite functions allows to prove that functions  $f, g, h$  satisfy equations

$$f_{x_1} g_{x_3} h_{x_2} + f_{x_2} g_{x_1} h_{x_3} = 0, \quad (5)$$

$$f_{x_2} g_{x_3} h_{x_4} = f_{x_4} g_{x_3} h_{x_2} + f_{x_2} g_{x_4} h_{x_3},$$

$$f_{x_{14}} g_{x_{34}} h_{x_{24}} + f_{x_{24}} g_{x_{14}} h_{x_{34}} = 0, \quad (6)$$

$$f_{x_{24}} g_{x_{34}} h_{x_4} = f_{x_4} g_{x_{34}} h_{x_{24}} + f_{x_{24}} g_{x_4} h_{x_{34}}.$$

This is equivalent to the degeneration of Jacobi matrices:

$$\text{rank} \begin{pmatrix} f_{x_1} & f_{x_2} & 0 & f_{x_4} \\ g_{x_1} & 0 & g_{x_3} & g_{x_4} \\ 0 & h_{x_2} & h_{x_3} & h_{x_4} \end{pmatrix} \leq 2,$$

$$\text{rank} \begin{pmatrix} f_{x_{14}} & f_{x_{24}} & 0 & f_{x_4} \\ g_{x_{14}} & 0 & g_{x_{34}} & g_{x_4} \\ 0 & h_{x_{24}} & h_{x_{34}} & h_{x_4} \end{pmatrix} \leq 2.$$

The first condition means that if we solve equations  $x_{12} = f$ ,  $x_{13} = g$  w.r.t.  $x_1, x_2$ , then the substitution into equation  $x_{23} = h$  cancels  $x_3, x_4$  identically and we come to some equation (4). Analogously, the second condition implies (3). ■

Thus, we see that the 4-th direction is actually on equal footing with the other ones and moreover, the odd/even sublattices carry an equation of Hirota type as well. The picture becomes completely symmetric if we consider the embedding  $\mathbb{Z}^4 \rightarrow \mathbb{Z}^5$  accordingly to the rule  $x_i \rightarrow x_{i5}$ .

## Definition of consistent quintuple. Equations

$$\langle 5 \rangle \quad f(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = 0,$$

$$\langle 4 \rangle \quad g(x_{12}, x_{13}, x_{15}, x_{23}, x_{25}, x_{35}) = 0,$$

$$\langle 3 \rangle \quad h(x_{12}, x_{14}, x_{15}, x_{24}, x_{25}, x_{45}) = 0,$$

$$\langle 2 \rangle \quad k(x_{13}, x_{14}, x_{15}, x_{34}, x_{35}, x_{45}) = 0,$$

$$\langle 1 \rangle \quad l(x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}) = 0$$

are called consistent if any three imply the other two **and** the same is true also for the shifted equations

$$T_5 \langle 5 \rangle \quad f(x_{125}, x_{135}, x_{145}, x_{235}, x_{245}, x_{345}) = 0,$$

$$T_4 \langle 4 \rangle \quad g(x_{124}, x_{134}, x_{145}, x_{234}, x_{245}, x_{345}) = 0,$$

$$T_3 \langle 3 \rangle \quad h(x_{123}, x_{134}, x_{135}, x_{234}, x_{235}, x_{345}) = 0,$$

$$T_2 \langle 2 \rangle \quad k(x_{123}, x_{124}, x_{125}, x_{234}, x_{235}, x_{245}) = 0,$$

$$T_1 \langle 1 \rangle \quad l(x_{123}, x_{124}, x_{125}, x_{134}, x_{135}, x_{145}) = 0.$$

## Another example: Desargues configuration

Let

$$H(a, b, c, d, e, f) = \frac{(a - b)(c - d)(e - f)}{(b - c)(d - e)(f - a)}.$$

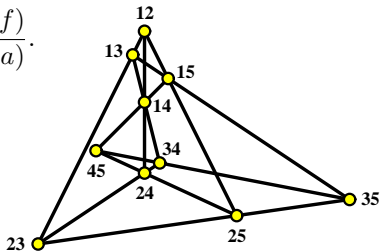
Equations

$$H(x_{ij}, x_{ik}, x_{kj}, x_{kl}, x_{jl}, x_{il}) = -1,$$

$$i, j, k, m \in \{1, 2, 3, 4, 5\}$$

are consistent.

Geometrically, each equation expresses Menelaus theorem and the whole consistent quintuple is contained in Desargues configuration.



[3] B.G. Konopelchenko, W.K. Schief. *J. Phys. A* **35**:29 (2002) [6125–6144](#).

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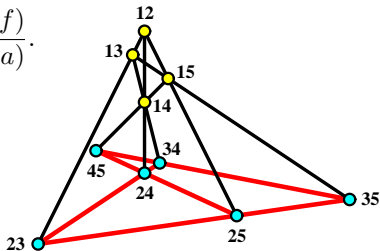
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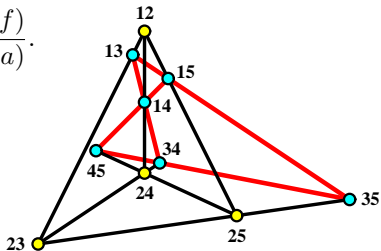
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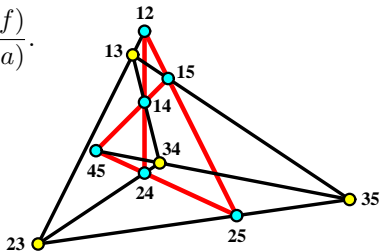
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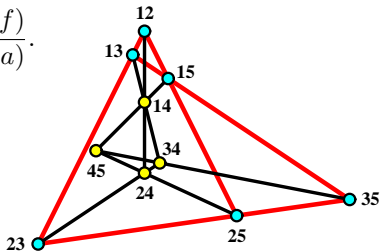
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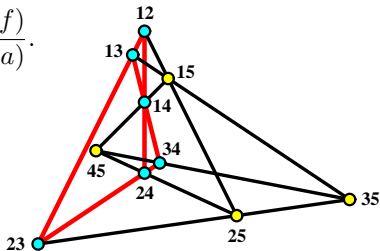
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## Classification theorem

Any 4D-consistent irreducible nonlinear autonomous equations of Hirota type is equivalent, up to nonautonomous point transformations, to one of the following:

$$x_{12}x_3 + x_{13}x_2 + x_{23}x_1 = 0 \quad (\chi_1)$$

$$(x_{13} - x_{12})x_1 + (x_{12} - x_{23})x_2 + (x_{23} - x_{13})x_3 = 0 \quad (\chi_2)$$

$$\frac{x_{13} - x_{12}}{x_1} + \frac{x_{12} - x_{23}}{x_2} + \frac{x_{23} - x_{13}}{x_3} = 0 \quad (\chi'_2)$$

$$\frac{(x_{12} - x_{13})(x_{23} - x_3)(x_2 - x_1)}{(x_{13} - x_{23})(x_3 - x_2)(x_1 - x_{12})} = -1 \quad (\chi_3)$$

$$\frac{x_{13} - x_{23}}{x_3} = x_{12} \left( \frac{1}{x_2} - \frac{1}{x_1} \right) \quad (\chi_4)$$

More precisely, all possible consistent quintuples, up to point transformations and permutations of indices are:

$$5(\chi_1) \quad (1 \leq i < j < k < l \leq 5)$$

$$x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0;$$

$$4(\chi_2) + (\chi_3) \quad (i, j, k \in \{1, 2, 3, 4\}):$$

$$(x_{ik} - x_{ij})x_{i5} + (x_{ij} - x_{jk})x_{j5} + (x_{jk} - x_{ik})x_{k5} = 0,$$

$$H(x_{12}, x_{13}, x_{23}, x_{34}, x_{24}, x_{14}) = -1;$$

$$4(\chi'_2) + (\chi_3) \quad (i, j, k \in \{1, 2, 3, 4\}):$$

$$\frac{x_{ik} - x_{ij}}{x_{i5}} + \frac{x_{ij} - x_{jk}}{x_{j5}} + \frac{x_{jk} - x_{ik}}{x_{k5}} = 0,$$

$$H(x_{12}, x_{13}, x_{23}, x_{34}, x_{24}, x_{14}) = -1;$$

**5( $\chi_3$ )** ( $i, j, k, m \in \{1, 2, 3, 4, 5\}$ ):

$$H(x_{ij}, x_{ik}, x_{kj}, x_{km}, x_{jm}, x_{im}) = -1;$$

**3( $\chi_4$ )+2( $\chi_2$ )** ( $i, j = 1, 2, 3$ ):

$$\frac{x_{i4} - x_{j4}}{x_{45}} = x_{ij} \left( \frac{1}{x_{j5}} - \frac{1}{x_{i5}} \right),$$

$$\frac{x_{13} - x_{12}}{x_{15}} + \frac{x_{12} - x_{23}}{x_{25}} + \frac{x_{23} - x_{13}}{x_{35}} = 0,$$

$$\frac{x_{14} - x_{24}}{x_{12}} + \frac{x_{24} - x_{34}}{x_{23}} + \frac{x_{34} - x_{14}}{x_{13}} = 0.$$

## Remarks

- We **assume** that **each** equation in the consistent quintuple is irreducible. This means that it is not of the form  $ab = 0$  where  $a$  and  $b$  depend on incomplete sets of variables.
- We **do not assume** that equations are polynomial or rational.
- However, we **assume** that equations are analytic in some domain and can be solved with respect to each variable. This eliminates tropical equations which are piece-wise linear.
- In contrast to 2D case, 3D equations do not contain essential parameters. All parameters can be eliminated by nonautonomous point changes, like the scaling we have used for Hirota equation:

$$x(m, n, k) \rightarrow a^{mn} b^{nk} c^{mk} x(m, n, k).$$

Of course, the choice of parameters must be consistent when we consider a set of five equations rather than a single one. For example, it is not possible to get all plus signs in all copies of  $(\chi_1)$ .



- All equations from the list can be derived from the auxiliary linear problems like

$$\psi_2 - \psi = u(\psi_1 - \psi), \quad \psi_3 - \psi = v(\psi_1 - \psi)$$

and are related to each other via difference substitutions. So, our main result can be reformulated as follows:

**The list of Hirota-type integrable equations is exhausted by Hirota equation itself and its three modifications.**

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The proof is sketched in the rest of the talk. The main tool is

**the three-leg form of equation.**

## Three-leg forms of Hirota-type equations

A more precise version of **Theorem 1** allows to make some statements on the form of consistent equations.

**Theorem 2.** If the triple (1) is consistent then it can be cast into the form

$$\begin{aligned} a(x_1, x_4, x_{14}) - b(x_2, x_4, x_{24}) &= p(x_{12}, x_{14}, x_{24}), \\ c(x_3, x_4, x_{34}) - a(x_1, x_4, x_{14}) &= q(x_{13}, x_{14}, x_{34}), \\ b(x_2, x_4, x_{24}) - c(x_3, x_4, x_{34}) &= r(x_{23}, x_{24}, x_{34}) \end{aligned}$$

and simultaneously into the form

$$\begin{aligned} A(x_1, x_4, x_{14}) - B(x_2, x_4, x_{24}) &= P(x_1, x_2, x_{12}), \\ C(x_3, x_4, x_{34}) - A(x_1, x_4, x_{14}) &= Q(x_1, x_3, x_{13}), \\ B(x_2, x_4, x_{24}) - C(x_3, x_4, x_{34}) &= R(x_2, x_3, x_{23}). \end{aligned}$$

Obviously, equations (3) and (4) are obtained now by summation.

**Sketch of the proof.** The first representation follows from solving the overdetermined eqs (5). First eq. (5) implies

$$\frac{f_1}{f_2} = -\frac{g_1/g_3}{h_2/h_3} = -\frac{\alpha(x_1, x_4, x_{14})}{\beta(x_2, x_4, x_{23})}$$

(it is sufficient to let  $x_3 = \text{const}$  in the middle expression). Now,

$$\frac{g_1}{\alpha g_3} = \frac{h_2}{\beta h_3} = -\gamma(x_3, x_4, x_{34})$$

where  $\gamma$  denotes the common value of both ratios. Therefore,

$$\beta f_1 + \alpha f_2 = 0, \quad \alpha g_3 + \gamma g_1 = 0, \quad \gamma h_2 + \beta h_3 = 0$$

and denoting  $\alpha = a_1$ ,  $\beta = b_2$ ,  $\gamma = c_3$  we prove

$$f = \phi(a(x_1, x_4, x_{14}) - b(x_2, x_4, x_{24}), x_4, x_{14}, x_{24})$$

and analogously for  $g, h$ . The use of the second eq. (5) allows to prove that  $\phi$  does not depend on  $x_4$ . The use of (6) allows to obtain the second representation analogously. ■

## Three-leg forms as a mechanism of consistency

Due to the symmetry of all coordinates in  $\mathbb{Z}^4$ , several other three-leg representations exist. For example, the permutations  $1 \leftrightarrow 4$  and  $2 \leftrightarrow 4$  in first equation from the previous Theorem:

$$a(x_1, x_4, x_{14}) - b(x_2, x_4, x_{24}) = p(x_{12}, x_{14}, x_{24})$$

provide two other equivalent forms for this equation,

$$\tilde{a}(x_1, x_4, x_{14}) - \tilde{b}(x_1, x_2, x_{12}) = \tilde{p}(x_{12}, x_{14}, x_{24}),$$

$$\bar{a}(x_1, x_2, x_{12}) - \bar{b}(x_2, x_4, x_{24}) = \bar{p}(x_{12}, x_{14}, x_{24}).$$

Next, we can solve equations (1) with respect to  $x_4$  and apply the shifts  $(T_i T_j T_4)^{-1}$ ,  $i, j = 1, 2, 3$ . This effectively changes the directions in the lattice, for example the first equation becomes

$$x_{-1, -2} = \tilde{f}(x_{-1}, x_{-2}, x_{-4}, x_{-1, -4}, x_{-2, -4}).$$

Applying **Theorem 2** to the corresponding triple of equations yields similar representations with the complementary subscripts  $(i, j, k) \leftrightarrow (jk, ik, ij)$ .

Thus,

**each equation under consideration admits eight equivalent three-leg representations**

and in total we have

**40 three-leg representations**

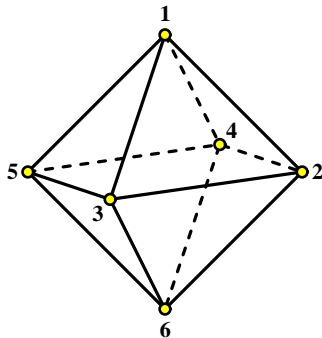
which, of course, must be consistent with each other.

Our strategy will be

- first, to analyze the three-leg forms of a single equation;
- next, to assemble these forms into a consistent quintuple.

## From consistent quintuple to a single equation

Let us consider just one member of consistent quintuple. In this context, we associate the variables with the vertices of an octahedron enumerated in such a way that  $i$  and  $I = 7 - i$  correspond to the opposite vertices.

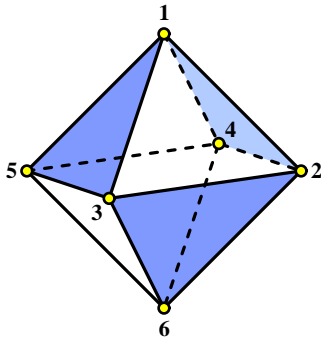


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An intrinsic property of any such equation is that it can be represented in eight equivalent forms as follows:

$$(123) \quad a^{142} + a^{263} + a^{351} = 0$$





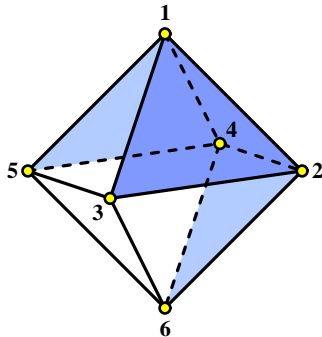
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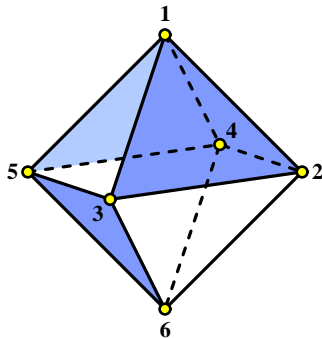
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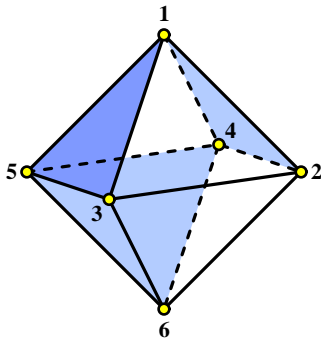
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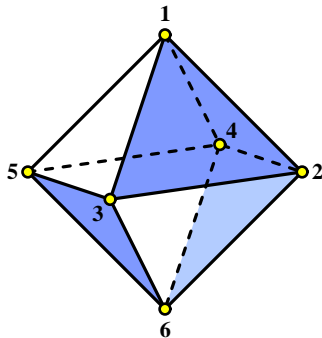
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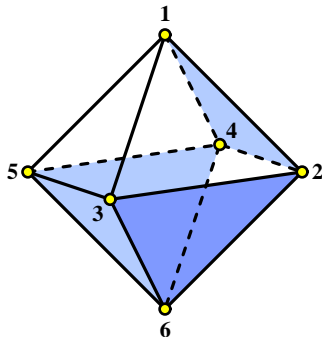
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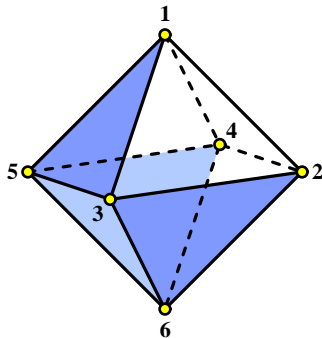
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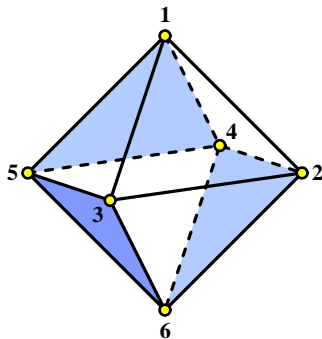
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$$(456) \quad a^{415} + a^{536} + a^{624} = 0$$



We call an equation with this property **three-leg** equation

## Classification of three-leg equations

Is this definition strict enough? **Yes**, only a finite list of three-leg equations exist.

**Theorem 3.** Three-leg equations are exhausted, up to the point changes  $x_i \rightarrow X_i(x_i)$  and the numeration of the vertices, by the following ones:

$$x_1x_6 + x_2x_5 + x_3x_4 = 0, \quad (\text{Y}_1)$$

$$(x_1 - x_2)x_4 + (x_2 - x_3)x_6 + (x_3 - x_1)x_5 = 0, \quad (\text{Y}_2)$$

$$\frac{(x_1 - x_4)(x_2 - x_6)(x_3 - x_5)}{(x_4 - x_2)(x_6 - x_3)(x_5 - x_1)} = -1, \quad (\text{Y}_3)$$

$$x_1x_6 = (x_2 + x_3)^{-\gamma}(x_4 + x_5), \quad (\text{Y}_4)$$

$$x_1x_6 = x_2 + x_3 + x_4 + x_5, \quad (\text{Y}_5)$$

$$x_1x_2x_3x_4 = x_5 + x_6, \quad (\text{Y}_6)$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0. \quad (\text{Y}_7)$$



The proof of this theorem is rather lengthy, although quite elementary. The main role play the identities (subscripts denote derivatives here)

$$\frac{a_i^{ikj} + a_i^{KJi}}{a_J^{KJi}} = \frac{a_i^{iKj} + a_i^{kJi}}{a_J^{kJi}}, \quad \frac{a_j^{ikj} + a_j^{jIK}}{a_I^{jIK}} = \frac{a_j^{iKj} + a_j^{jIk}}{a_I^{jIk}}$$

which can be easily obtained for any pair of three-leg forms with an edge  $(ij)$  in common:

$$\begin{aligned} (ijK) : \quad a^{ikj} + a^{jIK} + a^{KJi} &= 0, \\ (ijk) : \quad a^{iKj} + a^{jIk} + a^{kJi} &= 0. \end{aligned}$$

Notice that each of these equalities contains only 5 variables and therefore it must hold identically (not in virtue of the equation).

As a corollary, we obtain the identities

$$a_{ij}^{ikj} a_J^{kji} = a_{ij}^{iKj} a_J^{Kji}, \quad a_{ij}^{ikj} a_I^{jIk} = a_{ij}^{iKj} a_I^{jIK}$$

which allow to reduce the problem to functions depending on **two** variables.

**Statement.** The functions  $a^{ikj}$  and  $a^{iKj}$  are of the form

$$\begin{aligned} a^{ikj} &= a(x_i, x_j)b(x_k) + p(x_i, x_k) + q(x_k, x_j), \\ a^{iKj} &= a(x_i, x_j)c(x_K) + r(x_i, x_K) + s(x_K, x_j). \end{aligned}$$

The further analysis of the identities splits in many branches, but eventually it allows to determine all  $a^{ikj}$  up to point transformations.

## From single equation to consistent quintuple

Some combinations of three-leg equations are inconsistent just because the legs do not match. The following table lists all legs types, up to point transforms. For example, it implies that an equation of the type  $(Y_1)$  can be consistent only with equations of types  $(Y_1)$  or  $(Y_6)$ .

eq.	legs $a(x, y, z)$
$(Y_1)$	$xyz$
$(Y_2)$	$y(x+z), \log(x+y), \log\left(\frac{x+y}{y+z}\right)$
$(Y_3)$	$\log\left(\frac{x+y}{y+z}\right)$
$(Y_4)$	$y, xy, \log(x+y), y(x+z)^\gamma, y(x+z)^{1/\gamma}$
$(Y_5)$	$y, (x+y)z$
$(Y_6)$	$xyz, xy, y, y + \log(x+z), \log(x+y)$
$(Y_7)$	$y$

More precise results can be obtained by applying the **Theorem 2** which states that consistent equations can be brought to the form

$$\begin{array}{rcl}
 & & \langle m \rangle \\
 \langle i \rangle & [jn, jm, mn] - [kn, km, mn] & = [jn, jk, kn] \\
 \langle j \rangle & [kn, km, mn] - [in, im, mn] & = [kn, ki, in] \\
 \langle k \rangle & [in, im, mn] - [jn, jm, mn] & = [in, ij, jn]
 \end{array}$$

for any permutation  $(i, j, k, m, n) = \sigma(1, 2, 3, 4, 5)$ . Here the brackets denote, for short, the functions of three variables  $x$  with the corresponding double subscripts.

In particular, this allows to prove that equations of types  $(Y_4)$  at  $\gamma \neq 1$ ,  $(Y_5)$  and  $(Y_6)$  cannot be consistent at all. No quintuple exists which contain one equation of these types.

In the other cases, we find the form of equations up to 10 arbitrary functions  $X_{ij} = X_{ij}(x_{ij})$ , for example the quintuple of equations

$$H(X_{ij}, X_{ik}, X_{kj}, X_{km}, X_{jm}, X_{im}) = -1, \quad i, j, k, m \in \{1, 2, 3, 4, 5\}$$

possesses the above representation for any functions  $X_{ij}$ .

The final step consists of plugging these systems into the second set of consistent three-leg forms:

$$\begin{array}{rcl}
 & & T_m \langle m \rangle \\
 T_i \langle i \rangle & T_i([km, kn, kj] - [jm, jn, jk] = [km, mn, jm]) \\
 T_j \langle j \rangle & T_j([im, in, ik] - [km, kn, ki] = [im, mn, km]) \\
 T_k \langle k \rangle & T_k([jm, jn, ji] - [im, in, ij] = [jm, mn, im])
 \end{array}$$

This allows us to fix the functions  $X_{ij}$ . It turns out that in all cases these functions are related with each other via some linear-fractional transform (or just by scaling, as in case of  $(\chi_1)$ ). Moreover, all coefficients can be killed by the use of nonautonomous point changes and finally we come to the **classification theorem**.