Multidimensional quadrilateral lattices with the values in Grassmann manifold are integrable

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- Grassmann generalization of Darboux lattice
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Introduction: some 3D discrete integrable models (without reductions)

	dimension				
	vertex	edge	face	cube	hypercube
Q-net [1]	$\boxed{0}$	1	$\boxed{2}$	3	4
Grassmann Q-net	$\boxed{r}$	$2r+1$	$\boxed{3r+2}$	$4r+3$	$5r+4$
Darboux lattice [2, 3]	—	$\boxed{0}$	$\boxed{1}$	2	3
Grassmann-Darboux	—	$\boxed{r}$	$\boxed{2r+1}$	$3r+2$	$4r+3$
Line congruence [4, 5]	$\boxed{1}$	$\boxed{2}$	3	4	5

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[1] A. Doliwa, P.M. Santini. Multidimensional quadrilateral lattices are integrable. *Phys. Lett. A* **233:4–6** (1997) 365–372.

[2] W.K. Schief. *J. Nonl. Math. Phys.* **10:2** (2003) 194–208.

[3] A.D. King, W.K. Schief. *J. Phys. A* **39:8** (2006) 1899–1913.

[4] A. Doliwa, P.M. Santini, M. Mañas. *J. Math. Phys.* **41** (2000) 944–990.

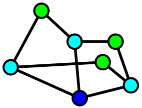
[5] A. Doliwa. *J. of Geometry and Physics* **39** (2001) 9–29.

Multidimensional quadrilateral lattices

A mapping $\mathbb{Z}^N \to \mathbb{P}^d$ is called N-dimensional Q-net if the vertices of any elementary cell are coplanar.

Main properties:

- 3-dimensional lattice is uniquely defined by three 2-dimensional ones;



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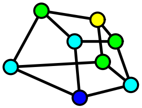

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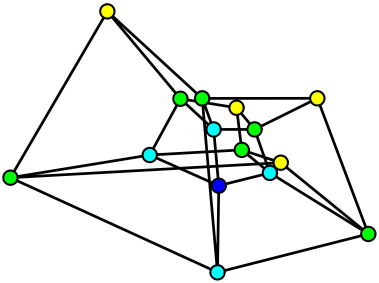

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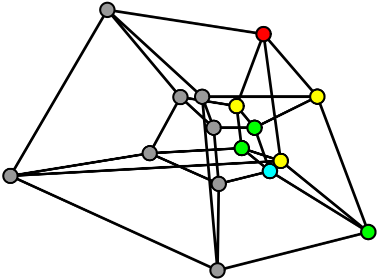

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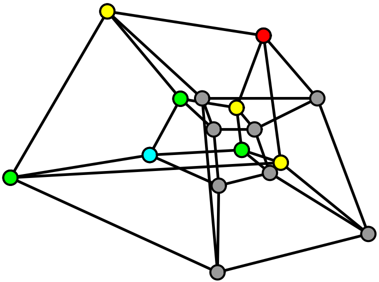

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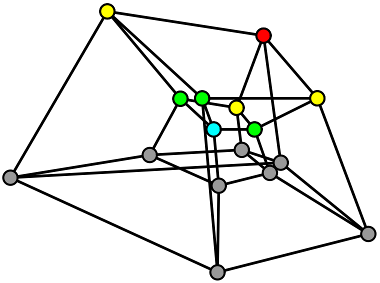

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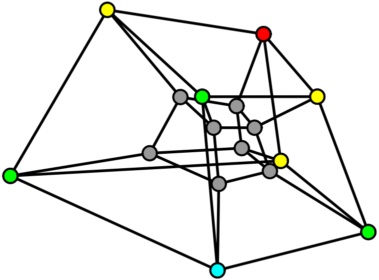

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Grassmann generalization of Q-nets

Recall that the Grassmann manifold G^{d+1}_{r+1} is defined as the variety of $(r+1)$-dimensional linear subspaces of some $(d+1)$-dimensional linear space.

Definition 1. A mapping

$$\mathbb{Z}^N \to G^{d+1}_{r+1}, \quad N \geq 2, \quad d > 3r+2,$$

is called the N-dimensional Grassmann Q-net of rank r, if any elementary cell maps to four r-dimensional subspaces in $\mathbb{P}^d$ which lie in a $(3r+2)$-dimensional one.

In other words, the images of any three vertices of a square cell are generic subspaces and their span contains the image of the last vertex.

We should check that:

- the initial data on three 2-dimensional coordinate planes in $\mathbb{Z}^3$ define a 3-dimensional Grassman Q-net;

- the initial data on six 2-dimensional coordinate planes in $\mathbb{Z}^4$ are not overdetermined and correctly define a 4-dimensional Grassman Q-net.

The proof of both properties will be based on the calculation of dimensions of subspaces,

$$\dim(A+B) = \dim A + \dim B - \dim(A \cap B).$$

Theorem 1. Let seven r-dimensional subspaces X, X_i, X_{ij}, $1 \leq i \neq j \leq 3$ be given in $\mathbb{P}^d$, $d \geq 4r+3$, such that

$$\dim(X + X_i + X_j + X_{ij}) = 3r + 2$$

for each pair of indices, but with no other degeneracies. Then the conditions

$$\dim(X_i + X_{ij} + X_{ik} + X_{123}) = 3r + 2$$

define an unique r-dimensional subspace X_{123}.

Proof. All subspaces under consideration lie in the ambient $(4r+3)$-dimensional space spanned over X, X_1, X_2, X_3. Generically, the subspaces $X_i + X_{ij} + X_{ik}$ are also $(3r+2)$-dimensional. The subspace X_{123}, if exists, lies in the intersection of three such subspaces. In the $(4r+3)$-dimensional space, the dimension of a pairwise intersection is $2(3r+2) - (4r+3) = 2r+1$, and therefore the dimension of the triple intersection is $(4r+3) - 3(3r+2) + 3(2r+1) = r$ as required. ■

Theorem 2. The 3-dimensional Grassmann Q-nets are 4D-consistent.

Proof. We have to check that six $(3r+2)$-dimensional subspaces through X_{ij}, X_{ijk}, X_{ijl} meet in a r-dimensional one (which is X_{1234}). This is equivalent to the computation of the dimension of intersection of four generic $(4r+3)$-dimensional subspaces in a $(5r+4)$-dimensional space which is r. ■

Discrete Darboux-Zakharov-Manakov system

Recall that the Grassmann manifold can be defined as

$$G^{d+1}_{r+1} = (V^{d+1})^{r+1}/GL_{r+1}$$

where GL_{r+1} acts as the base changes in any $(r+1)$-dimensional subspace of V^{d+1}. Such subspaces are identified with $(r+1)\times(d+1)$ matrices which are equivalent modulo left multiplication by matrices from GL_{r+1}.

We adopt the "affine" normalization by choosing the representatives as

$$x = \begin{pmatrix} x^{1,1} & \dots & x^{1,d-r} & 1 & \dots & 0 \\ \vdots & & \vdots & & \ddots & \\ x^{r+1,1} & \dots & x^{r+1,d-r} & 0 & \dots & 1 \end{pmatrix}.$$

Then the condition that the subspace X_{ij} belongs to the $(3r+2)$-dimensional linear span $X + X_i + X_j$ gives the following auxiliary linear problem with the matrix coefficients [6, 7]

$$x_{ij} = x + a^{ij}(x_i - x) + a^{ji}(x_j - x). \tag{1}$$

The calculation of the consistency conditions: one has to substitute x_{ik} and x_{jk} into

$$x_{ijk} = x_k + a_k^{ij}(x_{ik} - x_k) + a_k^{ji}(x_{jk} - x_k)$$

and to compare the results after permutation of i, j, k. This leads, in principle, to a birational map

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[6] L.V. Bogdanov, B.G. Konopelchenko. Lattice and q-difference Darboux-Zakharov-Mañakov systems via $\bar{\partial}$-dressing method. *J. Phys. A* **28:5** (1995) L173–178.

[7] A. Doliwa. Geometric algebra and quadrilateral lattices. arXiv: 0801.0512.

$$(a^{12}, a^{21}, a^{13}, a^{31}, a^{23}, a^{32}) \mapsto (a^{12}_3, a^{21}_3, a^{13}_2, a^{31}_2, a^{23}_1, a^{32}_1),$$

but it is too bulky even in the commutative case. Some change of variables is needed.

The consistency conditions imply, in particular, the relations

$$a^{ij}_k a^{ik} = a^{ik}_j a^{ij}. \tag{2}$$

This allows to introduce the discrete Lamé coefficients h^i by the formula

$$a^{ij} = h^i_j (h^i)^{-1}.$$

Now the linear problem takes the form

$$x_{ij} = x + h^i_j (h^i)^{-1}(x_i - x) + h^j_i (h^j)^{-1}(x_j - x)$$

and then one more change

$$x_i - x = h^i y^i, \qquad b^{ij} = (h^i_j)^{-1}(h^j_i - h^j)$$

brings it to the form

$$y^i_j = y^i - b^{ij} y^j. \tag{3}$$

The matrices b^{ij} are called the discrete rotation coefficients.

The compatibility conditions of the linear problems (3) are perfectly simple. We have

$$y^i_{jk} = y^i + b^{ik} y^k + b^{ij}_k (y^j + b^{jk} y^k) = y^i + b^{ij} y^j + b^{ik}_j (y^k + b^{kj} y^j)$$

which leads to the coupled equations

$$b^{ij}_k - b^{ik}_j b^{kj} = b^{ij}, \quad -b^{ij}_k b^{jk} + b^{ik}_j = b^{ik}$$

and finally to an explicit mapping.

Theorem 3. The compatibility conditions of equations (3) are equivalent to the birational mapping for the discrete rotation coefficients

$$b^{ij}_k = (b^{ij} + b^{ik} b^{kj})(I - b^{jk} b^{kj})^{-1}, \quad b^{ij} \in \mathrm{Mat}(r+1, r+1)$$

which is multi-dimensionally consistent.

Darboux lattice

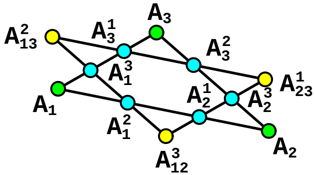
The lattice proposed in [2, 3] is a mapping

$$E(\mathbb{Z}^N) \quad \to \quad \mathbb{P}^d$$

such that the image of the edges of any elementary quadrilateral is a set of four collinear points.

Intersections of a fixed hyperplane with the lines corresponding to the edges of a Q-net form a Darboux lattice.

The picture demonstrates the images of a cube and a hypercube.



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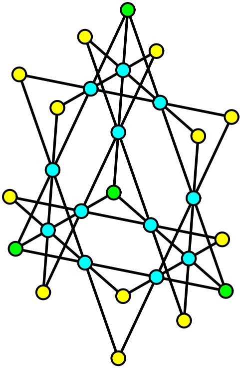

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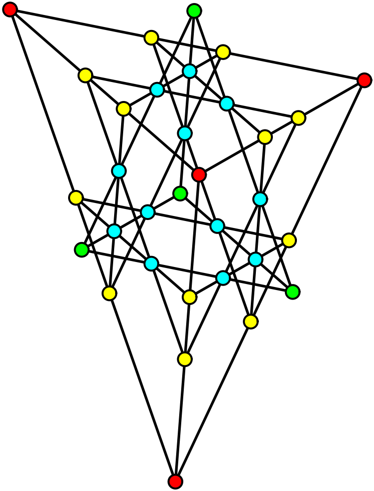


The Grassmann generalization of Darboux lattice

Definition 2. A mapping

$$E(\mathbb{Z}^N) \quad \to \quad G^{d+1}_{r+1}$$

is called Grassmann-Darboux lattice if the image of any elementary quadrilateral consists of four r-dimensional subspaces in $\mathbb{P}^d$ which lie in a $(2r+1)$-dimensional one.

As in $r = 0$ case, Grassmann-Darboux lattice is obtained from a Grassmann Q-net by intersection of some fixed subspace of codimension $r + 1$.

Let us demonstrate how to reduce Definition 2 to the discrete Darboux-Zakharov-Manakov system again. As before, we use the "affine" normalization, then

$$x^i_j = r^{ij}x^i + (I - r^{ij})x^j.$$

The consistency condition is

$$x^i_{jk} = r^{ij}_k(r^{ik}x^i + (I - r^{ik})x^k) + (I - r^{ij}_k)(r^{jk}x^j + (I - r^{jk})x^k)$$

and alteration of j, k yields

$$r^{ij}_k r^{ik} = r^{ik}_j r^{ij} \quad \Rightarrow \quad r^{ij} = s^i_j(s^i)^{-1}.$$

Now the change $(s^i)^{-1}x^i = y^i$ brings the linear problem to the form (3)

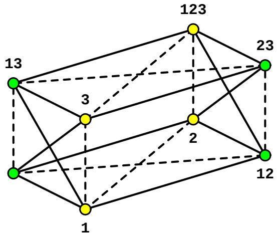
$$y^i_j = y^i - b^{ij}y^j, \quad b^{ij} = ((s^i)^{-1} - (s^i_j)^{-1})s^j.$$

Pappus vs. Moutard — 1:0

Recall that in the scalar case we have a plenty of reductions: reduction on quadric, orthogonal nets, Carnot reduction, A-nets, ..., Z-nets, ...

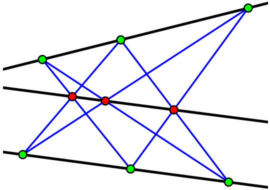
Do their analogs exist in the Grassmann case? This question maybe rather difficult to answer. No good examples are known for now.

The so-called Koenigs reduction of Q-nets seems to be a very natural candidate for the Grassmann generalization since it can be formulated in terms of subspaces: each set of four points x, x_{12}, x_{13}, x_{23} and x_1, x_2, x_3, x_{123} is coplanar (dashed lines).



[8] A.I. Bobenko, Yu.B. Suris. Discrete Koenigs nets and discrete isothermic surfaces. arXiv:0709.3408.

[9] A. Doliwa. Generalized isothermic lattices. *J. Phys. A* **40** (2007) 12539–12561.



A Grassmann generalization seems obvious, but meets an obstacle.

The explanation is that the existence of Koenigs reduction is based on the well known Möbius theorem on two mutually inscribed tetrahedra. This theorem is proved with the use of Pappus hexagram theorem which, in turn, is equivalent to the commutativity of the multiplication in the field of constants [10].

[10] D. Hilbert. Grundlagen der Geometrie. Leipzig, 1899.

The related example of Moutard reduction corresponds to the skew symmetry $a^{ij} = -a^{ji}$ of the coefficients in equation (1). Recall that this choice leads to such important integrable models as star-triangle map and discrete BKP equation.

In the noncommutative case, this reduction turns equations (2) into

$$a^{ij}_k a^{ki} = a^{ki}_j a^{ij}, \quad a^{jk}_i a^{ij} = a^{ij}_k a^{jk}, \quad a^{ki}_j a^{jk} = a^{jk}_i a^{ki}$$

which lead to the constraint

$$a^{ki}(a^{ij})^{-1}a^{jk} = a^{jk}(a^{ij})^{-1}a^{ki}.$$

Moreover, the constraints corresponding to eight cubes adjacent to a common vertex are not compatible with each other, so that the global construction of a lattice satisfying such constraint is not possible, cf [7].

Construction of Grassmann reductions remains an open problem.