KdV equation and Volterra lattice: negative flows and multicomponent Painlevé type reductions

V.E. Adler

L.D. Landau ITP

17 Nov 2023

Outline

- Stationary equations for non-autonomous symmetries
- Recursion operator and negative symmetries
- Systems of Painlevé type
- Bäcklund transformations

◦ V.A., M.P. Kolesnikov. J. Math. Phys. 64 (2023) 101505
◦ V.A. arXiv:2307.08127.

KdV equation $u_t = u_{xxx} - 6uu_x$

Recursion operator ($D = \partial_x$):

$$R = D^2 - 4u - 2u_x D^{-1}$$

Seed symmetries:

$$u_{t_0} = u_x, \quad u_{\tau_0} = 6tu_x - 1$$

Symmetry algebra:

$$u_{t_i} = R^i(u_x) = D(h_n), \qquad u_{\tau_i} = R^i(6tu_x - 1)$$

$$[\partial_{t_i}, \partial_{t_j}] = 0, \qquad [\partial_{\tau_i}, \partial_{t_j}] = (4j+2)\partial_{t_{j+i-1}}, \qquad [\partial_{\tau_i}, \partial_{\tau_j}] = 4(j-i)\partial_{\tau_{j+i-1}}$$

Novikov equation [Funct. An. Appl. 8:3, 1974]:

$$A[R](u_x) = a_0 u_{t_0} + a_1 u_{t_1} + \dots + a_n u_{t_n} = 0 \quad \longrightarrow \quad n\text{-gap solutions}$$

A complete set of first integrals, Liouville integrability, Dubrovin equations, solutions in theta-functions.

The problem

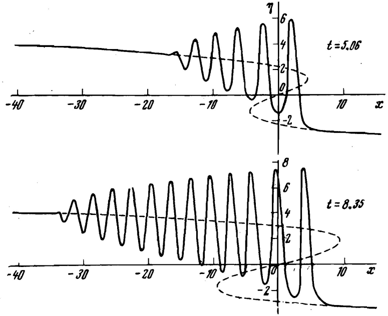
$$A[R](u_x) + B[R](6tu_x - 1) = 0 \quad \longrightarrow \quad ? \qquad (1)$$

Examples:

- $u_{t_1} + u_{\tau_0} = 0$. Galilean invariance, Painlevé-I equation $u_{zz} = 6u^2 + z$
- $u_{\tau_1} = 0$. Scaling invariance, P-II equation $f_{zz} = 2f^3 + zf + \alpha$
- $u_{t_2} + u_{\tau_0} = 0$. Fourth-order Painlevé type equation

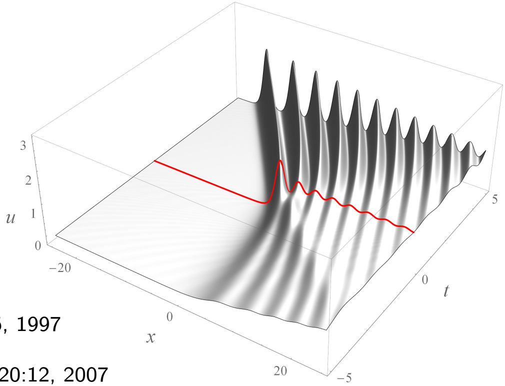
$$u_{xxxx} - 10uu_{xx} - 5u_x^2 + 10u^3 + 6tu + x = 0$$

◦ G. Moore. Comm. Math. Phys. 133:2, 1990
◦ B.I. Suleimanov. JETP 78:5, 1994
◦ V. Kudashev, B. Suleimanov. Phys. Lett. A 221:3, 1996
◦ B. Dubrovin. Comm. Math. Phys. 267, 2006
◦ A.V. Gurevich, L.P. Pitaevskii. JETP Lett. 17:5, 1973; JETP 38:2, 1974



- $u_{t_n} + u_{\tau_0} = 0$. Higher P-I equations
- $u_{t_n} + u_{\tau_1} = 0$. Higher P-II equations
- $u_{\tau_2} + au_{\tau_1} + bu_{\tau_0} = 0$. Sixth-order equation

$$3t(u_{xxxx} - 10uu_{xx} - 5u_x^2 + 10u^3)_x + x(u_{xxx} - 6uu_x) + 4u_{xx} - 8u^2 - 2u_x D^{-1}(u) + \\ + a(3t(u_{xxx} - 6uu_x) + xu_x + 2u) + b(6tu_x - 1) = 0$$



◦ N.A. Kudryashov. Phys. Lett. A 224:6, 1997
J. Phys. A 35, 2002
◦ M. Mazzocco, M.Y. Mo. Nonlinearity 20:12, 2007
◦ V.A. J. Nonl. Math. Phys. 27:3, 2020

Equivalent ODEs

Assume that

$$\deg A \leq \deg B = n, \quad B = (R + 4\alpha_1)\cdots(R + 4\alpha_n), \quad \alpha_i \neq \alpha_j.$$

Theorem 1

Equation (1) is equivalent to the non-autonomous ODE system

$$y_{j,xx} = \frac{y_{j,x}^2 - \beta_j^2}{2y_j} + 2(u - \alpha_j)y_j, \quad j = 1, \ldots, n, \tag{2}$$

$$u := \frac{x}{6t} + \frac{1}{3t}(y_1 + \cdots + y_n). \tag{3}$$

The derivation ∂_t defined by

$$y_{j,t} = 2u_x y_j - 2(u + 2\alpha_j)y_{j,x} \tag{4}$$

commutes with ∂_x defined by (2). The variable u defined by (3) satisfies the KdV eq. $u_t = u_{xxx} - 6uu_x$ in virtue of (2) and (4).

Remark. If $\deg A - \deg B = r > 0$ then (2) and (4) do not change, while the constraint (3) is replaced with

$$D^{-1}C[R](u_x) + 6tu - x = 2y_1 + \cdots + 2y_n, \quad \deg C = r.$$

Scheme of proof

$$A[R](u_x) + B[R](6tu_x - 1) = 0 \quad \Leftrightarrow$$

$$6tu_x - 1 = -B^{-1}A(u_x) = \sum_{j=1}^{n} \gamma_j (R + 4\alpha_j)^{-1}(u_x) = \sum_{j=1}^{n} 2y_{j,x} \qquad (5)$$

where γ_j are set to 2 without loss of generality and

$$y_{j,x} = (R + 4\alpha_j)^{-1}(u_x).$$

It remains to integrate (5) and to prove that y_j satisfy (2) and (4).

Let us consider this in more detail.

Negative symmetries

Definition

Let R be the recursion operator, then the flow

$$u_z = (R + 4\alpha)^{-1}(u_x) \tag{6}$$

is called negative symmetry.

For the KdV equation, the seed symmetry u_x is the factor at the integral term:

$$R = D^2 - 4u - 2u_x D^{-1} \quad \Rightarrow \quad R(f) \sim R(f) + Cu_x.$$

This means that (6) can be replaced with

$$R(u_z) = -4\alpha u_z.$$

Let $u_z = y$, then

$$y_{xxx} - 4uy_x - 2u_x y = -4\alpha y_x.$$

This can be integrated once more after multiplication by $2y$.

Negative symmetry of the KdV hierarchy

$$u_z = y_x \tag{7}$$

where y is non-local variable defined by equations

$$2yy_{xx} - y_x^2 - 4(u-\alpha)y^2 + \beta^2 = 0 \tag{8}$$

$$y_{t_n} = h_{n,z}.$$

Equation for y coincides with (2). Therefore, the constraint (3) is equivalent to the stationary equation for the sum of Galilean symmetry and negative flows corresponding to different parameters:

$$u_{\tau_0} - 2u_{z_1} - \cdots - 2u_{z_n} = 6tu_x - 1 - 2y_{1,x} - \cdots - 2y_{n,x} = 0.$$

Remarks

- (8) is the equation for the resolvent of the Sturm–Liouville operator.

◦ I.M. Gelfand, L.A. Dikii. Russ. Math. Surv. 30:5, 1975

- ∂_z can be viewed as the generating function for the flows ∂_{t_i}:

$$y = 1 + h_1/(-4\alpha) + h_2/(-4\alpha)^2 + \dots, \quad u_{t_i} = h_{i,x}.$$

- Since $[\partial_{t_i}, \partial_{t_j}] = 0$, the negative flows ∂_{z_i} corresponding to different $\alpha = \alpha_i$ commute as well.

- the change $u = v_x$, $y = v_z$ brings to

$$2v_z v_{xxz} - v_{xz}^2 - 4(v_x - \alpha)v_z^2 + \beta^2 = 0$$

which is point equivalent to the Camassa–Holm equation.

- The commutativity property $[\partial_{z_i}, \partial_{z_j}] = 0$ implies that v satisfies the 3D equation

$$(\alpha_i - \alpha_j)v_{z_i z_j} = v_{z_i} v_{xz_j} - v_{z_j} v_{xz_i}.$$

- J. Schiff. Phys. D 121, 1998
- A.N.W. Hone. J. Phys. A 32, 1999
- L. Martínez Alonso, A.B. Shabat. Phys. Lett. A 300:1, 2002
- V.A., A.B. Shabat. Theor. Math. Phys. 153:1, 2007

- Solution of (8) can be represented by squared eigenfunctions of the Sturm–Liouville operator:

$$y=\psi\varphi,\quad \psi_{xx}=(u-\alpha)\psi,\quad \varphi_{xx}=(u-\alpha)\varphi.$$

  This also give a method for deriving t-evolution (4) from equations

$$\psi_t=u_x\psi-2(u+2\alpha)\psi_x,\quad \varphi_t=u_x\varphi-2(u+2\alpha)\varphi_x.$$

- System (2) is equivalent to the non-autonomous version of the Garnier system:

$$\psi_{j,xx}=(u-\alpha_j)\psi_j,\quad \varphi_{j,xx}=(u-\alpha_j)\varphi_j,$$
$$u:=\frac{x}{6t}+\frac{1}{3t}(\psi_1\varphi_1+\dots+\psi_n\varphi_n).$$

◦ R. Garnier. Rend. Circ. Mathem. Palermo 43:4, 1919.
◦ D.V. Choodnovsky, G.V. Choodnovsky. Séminaire Polytechnique 6, 1978.
◦ A.Yu. Orlov, S. Rauch-Wojciechowski. Phys. D 69, 1993.

Lax pairs

KdV hierarchy is the compatibility conditions for

$$\Psi_x = U\Psi, \quad \Psi_{t_i} = V_i\Psi, \quad \Psi_{\tau_i} + (-4\lambda)^i\Psi_\lambda = W_i\Psi$$

where U, V_i and W_i are of the form

$$M(g) = \begin{pmatrix} -g_x & 2g \\ 2(u-\lambda)g - g_{xx} & g_x \end{pmatrix}.$$

In particular,

$$U = M(1/2) = \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix}, \quad V_1 = M(-u-2\lambda), \quad W_0 = M(3t).$$

Proposition

Equations (2)–(4) admit the isomonodromy Lax representations

$$A_x = U_\lambda + [U, A], \quad A_t = V_\lambda + [V, A]$$

where

$$A = M\left(-3t - \frac{1}{2}\Big(\frac{y_1}{\lambda-\alpha_1} + \dots + \frac{y_n}{\lambda-\alpha_n}\Big)\right).$$

Bäcklund transformations

Parameters β_j are changed by transformations $A_1, B_1, \dots, A_n, B_n$ (but all α_j are fixed, up to the obvious S_n symmetry).

Transformations A_k are trivial: they only change the sign of β_k without actual changing the equation:

$$A_k: \quad \begin{cases} \tilde{y}_j = y_j, \\ \tilde{\beta}_j = \beta_j, \quad j \neq k, \quad \tilde{\beta}_k = -\beta_k. \end{cases}$$

Transformation B_k is the Darboux transformation for $-D^2 + u$, which is defined by auxiliary function $f_k = \psi_{k,x}/\psi_k$ and it turns out that ψ_k must correspond to $\lambda = \alpha_k$:

$$B_k: \quad \begin{cases} f_k = \dfrac{y_{k,x} + \beta_k}{2y_k}, \\ \tilde{y}_j = \dfrac{(y_{j,x} - 2y_j f_k)^2 - \beta_j^2}{4(\alpha_j - \alpha_k) y_j}, \quad \tilde{\beta}_j = \beta_j, \quad j \neq k, \\ \tilde{y}_k = -y_k + 6t(f_k^2 + \alpha_k) - x - \displaystyle\sum_{j \neq k} (\tilde{y}_j + y_j), \quad \tilde{\beta}_k = 1 - \beta_k. \end{cases}$$

Proposition

Transformations A_j, B_k preserve the form of eqs. (2), (3), (4). These transformations are involutive and permutable with each other if $j \neq k$:

$$A_k^2 = \mathrm{id}, \quad A_j A_k = A_k A_j, \quad B_k^2 = \mathrm{id}, \quad B_j B_k = B_k B_j, \quad A_j B_k = B_k A_j, \quad j \neq k.$$

The group generated by $A_1, B_1, \dots, A_n, B_n$ is isomorphic to $\mathbb{Z}_2^n \times \mathbb{Z}^n$.

Volterra lattice $u_{n,x} = u_n(u_{n+1} - u_{n-1})$

Recursion operator

$$R = u_n + u_n(u_{n+1}T^2 - u_{n-1}T^{-1})(T-1)^{-1}\frac{1}{u_n}, \qquad T : n \mapsto n+1$$

Local part of the hierarchy

$$u_{n,t_i} = R^{i-1}(u_{n,x}) = u_n(T - T^{-1})(h_n^{(i)}), \quad i = 1, 2, \dots$$

Levi system ($x := t_1$, $t := t_2$)

$$u_{n,t} = u_n(T - T^{-1})\big(u_n(u_{n+1} + u_n + u_{n-1})\big)$$

$$\Updownarrow$$

$$u_t = -u_{xx} + (2uv + u^2)_x, \quad v_t = v_{xx} + (2uv + v^2)_x$$

◦ S.V. Manakov, JETP 40:2, 1975
◦ D. Levi, J. Phys. A 14:5, 1981
◦ W. Oevel, H. Zhang, B. Fuchssteiner, Progr. Theor. Phys. 81:2, 1989

Non-local part of the hierarchy

$$u_{n,\tau_j} = R^{j-1}(u_{n,\tau_1})$$

scaling $\quad u_{n,\tau} = u_{n,\tau_1} = u_n + xu_{n,x} + 2tu_{n,t} + 3t_3 u_{n,t_3} + \dots$

master-symmetry $\quad u_{n,\tau_2} = xu_{n,t} + u_n((n+3)u_{n+1} + u_n - nu_{n-1})$

Simplest Painlevé-type reductions

$u_{n,\tau_1} = 0$:

$$4tu_n(u_{n+1} + u_n + u_{n-1}) + 2xu_n + n - \delta + (-1)^n\varepsilon = 0 \qquad (\mathrm{dP}_1)$$

$u_{n,\tau_2} - \alpha u_{n,\tau_1} - \delta u_{n,x} = 0$:

$$(y_{n+1} + y_n)(y_n + y_{n-1}) = 2x\frac{\alpha y_n^2 + (-1)^n\beta y_n + \gamma_n}{y_n - n + \delta - (-1)^n\varepsilon} \qquad (\mathrm{dP}_{34})$$

◦ A.S. Fokas, A.R. Its, A.V. Kitaev, Comm. Math. Phys. 142, 1991
◦ V.A., A.B. Shabat, Theor. Math. Phys. 201, 2019
◦ B. Grammaticos, A. Ramani. Physica Scr. 89, 2014

Negative symmetry

Like in the continuous case, the general constraint

$$A[R](u_{n,x})+B[R](u_{n,\tau})=0$$

is equivalent (assuming that zeroes of B are simple) to

$$u_{n,\tau}=(R-\alpha^1)^{-1}(u_{n,x})+\dots+(R-\alpha^m)^{-1}(u_{n,x}).$$

Negative flow is defined by

$$u_{n,z}=g_n=(R-\alpha)^{-1}(u_{n,x})\quad\Leftrightarrow\quad(R-\alpha)(g_n)=u_n(u_{n+1}-u_{n-1}).$$

Since the term $(T-1)^{-1}$ in

$$R=u_n+u_n(u_{n+1}T^2-u_{n-1}T^{-1})(T-1)^{-1}\frac{1}{u_n}$$

adds an arbitrary integration constant, this can be replaced with

$$Rg_n=\alpha g_n.$$

Let $g_n = u_n(G_{n+1} - G_n)$, then

$$u_n^2(G_{n+1} - G_n) + u_n(u_{n+1}G_{n+2} - u_{n-1}G_{n-1}) = \alpha u_n(G_{n+1} - G_n) \quad \Rightarrow$$

$$(T+1)(u_nG_{n+1} - u_{n-1}G_{n-1}) - \alpha(G_{n+1} - G_n) = 0.$$

Now set $G_n = y_n + y_{n-1}$, then

$$u_n(y_{n+1} + y_n) - u_{n-1}(y_{n-1} + y_{n-2}) - \alpha(y_n - y_{n-1}) + (-1)^n\beta = 0.$$

This can be integrated once more after multiplication by $y_n + y_{n-1}$ and gives

negative symmetry of the Volterra hierarchy:

$$u_{n,z} = u_n(y_{n+1} - y_{n-1}) \tag{9}$$

where y_n is non-local variable defined by equations

$$u_n(y_{n+1} + y_n)(y_n + y_{n-1}) = \alpha y_n^2 + (-1)^n\beta y_n + \gamma \tag{10}$$

$$y_{n,t_i} = h^{(i)}_{n,z}$$

Remarks

- For $\beta = 0$ and $\gamma = -\alpha/4$, the negative flow is the generating function for ∂_{t_i}:

$$y_n = \frac{1}{2} + \frac{h_n^{(1)}}{\alpha} + \frac{h_n^{(2)}}{\alpha^2} + \dots, \quad u_{n,t_i} = u_n(T - T^{-1})(h_n^{(i)}).$$

  Eq. (10) is equivalent to explicit recurrence relations

$$h_n^{(i+1)} = u_n \sum_{s=0}^{i} \big(h_{n+1}^{(s)} + h_n^{(s)}\big)\big(h_n^{(i-s)} + h_{n-1}^{(i-s)}\big) - \sum_{s=1}^{i} h_n^{(s)} h_n^{(i+1-s)}, \quad h_n^{(0)} = \frac{1}{2}. \tag{11}$$

- Flows ∂_{z_i} corresponding to different $\alpha = \alpha_i$ commute as well.
- Up to simple transformations, there are three cases

$$\alpha \neq 0, \ \ \beta = 0; \qquad \alpha = \gamma = 0, \ \ \beta = 1; \qquad \alpha = \beta = 0, \ \ \gamma = 1.$$

  The last case is simplified (by the change $y_{n+1} + y_n = 1/p_n$) to

$$u_{n,z} = p_{n-1} - p_n, \quad u_n = p_{n-1} p_n$$

  which was studied in

  - G.M. Pritula, V.E. Vekslerchik. J. Phys. A 36:1, 2003
  - X.B. Hu, W.M. Xue. J. Phys. Soc. Japan 72:12, 2003

- For each n, variables $y_n = p$, $y_{n+1} = q$ and $u_n = u$, $u_{n+1} = v$ satisfy equations

$$\begin{cases} u_\xi = p_x = u(p+q) - \dfrac{\alpha p^2 + \hat{\beta} p + \gamma}{p+q}, \\ v_\xi = q_x = -v(p+q) + \dfrac{\alpha q^2 - \hat{\beta} q + \gamma}{p+q} \end{cases}$$

which define the negative symmetry for the Levi system.

Results

Theorem 2

Equation $A[R](u_{n,x})+B[R](u_{n,\tau})=0$ is equivalent to the non-autonomous OΔE system

$$u_n(y^j_{n+1}+y^j_n)(y^j_n+y^j_{n-1})=\alpha^j(y^j_n)^2+(-1)^n\beta^j y^j_n+\gamma^j,\quad j=1,\dots,m, \tag{12}$$

$$2\sum_{i=1}^{r} it_i h^{(i)}_n+n-\delta+(-1)^n\varepsilon=y^1_n+\dots+y^m_n \tag{13}$$

where $h^{(i)}_n$ are homogeneous polynomials on $u_{n-i+1},\dots,u_{n+i-1}$ defined by recurrent relations (11). This system is consistent with the derivations

$$u_{n,t_i}=u_n(h^{(i)}_{n+1}-h^{(i)}_{n-1}),\quad y^j_{n,t_i}=h^{(i)}_{n,z_j}.$$

For $r=1$, eq. (13) takes the form

$$u_n=\frac{1}{2x}\left(y^1_n+\dots+y^m_n-n+\delta-(-1)^n\varepsilon\right).$$

Let all $\alpha^j \neq 0$, $\beta^j = 0$ and $\gamma^j = -\alpha^j(\omega^j)^2$. Define the transformations

$$A_k: \quad \tilde{\omega}^k = -\omega^k;$$

$$B_k: \quad \begin{cases} \tilde{u}_n = u_n \dfrac{(y^k_{n+1} - \omega^k)(y^k_{n-1} + \omega^k)}{(y^k_n)^2 - (\omega^k)^2}, \quad \tilde{\omega}^k = 1 - \omega^k, \quad \tilde{\varepsilon} = -\varepsilon, \\ \tilde{y}^j_n = \dfrac{1}{\alpha^j - \alpha^k}\left((\alpha^j + \alpha^k)y^j_n - \alpha^k(y^k_n + \omega^k)\dfrac{y^j_{n+1} + y^j_n}{y^k_{n+1} + y^k_n}\right. \\ \qquad\qquad \left. - \alpha^k(y^k_n - \omega^k)\dfrac{y^j_n + y^j_{n-1}}{y^k_n + y^k_{n-1}}\right), \quad j \neq k, \\ \tilde{y}^k_n = -\displaystyle\sum_{j\neq k}\tilde{y}^j_n + 2\sum_{i=1}^{r} it_i h^{(i)}_n(\tilde{u}_{n-i+1}, \dots, \tilde{u}_{n+i-1}) + n - \delta - (-1)^n\varepsilon. \end{cases}$$

Proposition

Transformations A_k, B_k preserve eqs. (12), (13) and satisfy identities

$$A_k^2 = \mathrm{id}, \;\; A_jA_k = A_kA_j, \;\; B_k^2 = \mathrm{id}, \;\; B_jB_k = B_kB_j, \;\; A_jB_k = B_kA_j, \;\; j \neq k.$$

The group generated by $A_1, B_1, \dots, A_n, B_n$ is isomorphic to $\mathbb{Z}_2^n \times \mathbb{Z}^n$.