Painlevé-type reductions for the non-Abelian Volterra lattices

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Outline

Two non-Abelian (matrix) Volterra lattices

- $\mathrm{VL}^1 \quad u_{n,x} = u_{n+1}u_n - u_nu_{n-1}$ (M.A. Salle, Theor. Math. Phys. 1982)
- $\mathrm{VL}^2 \quad u_{n,x} = u^{\mathrm{T}}_{n+1}u_n - u_nu^{\mathrm{T}}_{n-1}$ (new, arXiv:2010.09021)

Results

- Substitutions $\mathrm{VL}^1 \leftarrow \cdots \rightarrow \mathrm{VL}^2$
- Symmetries
- Lax pairs
- Higher symmetry + scaling $\rightarrow \mathrm{dP}^i_1 + \mathrm{P}^i_4$
- Master-symmetry + scaling + D_x $\rightarrow \mathrm{dP}^i_{34} + \mathrm{P}^i_5$ $\qquad i = 1, 2$
- Master-symmetry + D_x $\rightarrow \mathrm{dP}^i_{34} + \mathrm{P}^i_3$

$\mathrm{VL}^1 \leftarrow \mathrm{mVL}^1 \leftarrow \text{pot-mVL} \rightarrow \mathrm{mVL}^2 \rightarrow \mathrm{mVL}^2$

The relation between VL^1 and mVL^2 is far from obvious.

$$\begin{aligned}
\mathrm{VL}^1: &\quad u_{n,x} = u_{n+1}u_n - u_n u_{n-1}\\
\mathrm{mVL}^1: &\quad v_{n,x} = v_{n+1}(v_n^2-\alpha^2) - (v_n^2-\alpha^2)v_{n-1} \qquad\qquad (\alpha\in\mathbb{C})\\
\text{pot-mVL}: &\quad w_{n,x} = (w_{n+1}+2\alpha w_n)(w_{n-1}^{-1}w_n+2\alpha)\\
\mathrm{mVL}^2: &\quad v_{n,x} = (v_n-\alpha)v_{n+1}(v_n+\alpha) - (v_n+\alpha)v_{n-1}(v_n-\alpha)\\
\mathrm{VL}^2: &\quad u_{n,x} = u^{\mathrm{T}}_{n+1}u_n - u_n u^{\mathrm{T}}_{n-1}
\end{aligned}$$

Substitutions:

$$\begin{aligned}
\mathrm{VL}^1 \leftarrow \mathrm{mVL}^1: &\quad u_n = (v_{n+1}+\alpha)(v_n-\alpha) \qquad \text{discrete Miura map}\\
\mathrm{mVL}^1 \leftarrow \text{pot-mVL}: &\quad v_n = w_{n+1}w_n^{-1}+\alpha\\
\text{pot-mVL} \rightarrow \mathrm{mVL}^2: &\quad v_n = w_n^{-1}w_{n+1}+\alpha\\
\mathrm{mVL}^2 \rightarrow \mathrm{VL}^2: &\quad \begin{cases} u_n = (v_n+\alpha)(v_{n-1}+\alpha) & \text{for even } n\\ u_n = (v^{\mathrm{T}}_n-\alpha)(v^{\mathrm{T}}_{n-1}-\alpha) & \text{for odd } n\end{cases}
\end{aligned}$$

Remark: an (incomplete) analogy with KdV

There is a sequence of substitutions

$$\mathrm{KdV} \xleftarrow[\text{Miura map}]{u=v^2\pm v_x+\alpha} \mathrm{mKdV}^1 \xleftarrow{v=w_xw^{-1}} \text{pot-mKdV} \xrightarrow{v=w^{-1}w_x} \mathrm{mKdV}^2$$

between

$$\begin{aligned} \mathrm{KdV}:&\quad u_t = u_{xxx} - 3uu_x - 3u_xu \\ \mathrm{mKdV}^1:&\quad v_t = v_{xxx} - 3v^2v_x - 3v_xv^2 - 6\alpha v_x \\ \text{pot-mKdV}:&\quad w_t = w_{xxx} - 3w_{xx}w^{-1}w_x - 6\alpha w_x \\ \mathrm{mKdV}^2:&\quad v_t = v_{xxx} + 3[v,v_{xx}] - 6vv_xv - 6\alpha v_x \end{aligned}$$

These equations can be obtained from the corresponding lattice equations by continuous limit, but no continuous analog of VL^2 is known.

Symmetries: basic derivations

- $D_x = D_{t_1}$, the lattice itself
- D_{t_2}, the simplest higher symmetry

$$\begin{aligned}\mathrm{VL}^1:\quad u_{n,t_2} &= (u_{n+2}u_{n+1} + u_{n+1}^2 + u_{n+1}u_n)u_n\\ &\qquad - u_n(u_nu_{n-1} + u_{n-1}^2 + u_{n-1}u_{n-2})\\ \mathrm{VL}^2:\quad u_{n,t_2} &= (u^{\mathrm{T}}_{n+1}u_{n+2} + (u^{\mathrm{T}}_{n+1})^2 + u_nu^{\mathrm{T}}_{n+1})u_n\\ &\qquad - u_n(u^{\mathrm{T}}_{n-1}u_n + (u^{\mathrm{T}}_{n-1})^2 + u_{n-2}u^{\mathrm{T}}_{n-1})\end{aligned}$$

- D_{τ_1}, the classical scaling symmetry

$$u_{n,\tau_1} = u_n$$

- D_{τ_2}, the master-symmetry (nonlocal for VL^1, local for VL^2)

$$\begin{aligned}\mathrm{VL}^1:\quad & u_{n,\tau_2} = \big(n+\tfrac{3}{2}\big)u_{n+1}u_n + u_n^2 - \big(n-\tfrac{3}{2}\big)u_nu_{n-1} + [s_n, u_n],\\ & s_n - s_{n-1} = u_n\\ \mathrm{VL}^2:\quad & u_{n,\tau_2} = \big(n+\tfrac{3}{2}\big)u^{\mathrm{T}}_{n+1}u_n + u_n^2 - \big(n-\tfrac{3}{2}\big)u_nu^{\mathrm{T}}_{n-1}\end{aligned}$$

Remark: associated systems

Due to the lattice, any variable u_{n+k} is an expression of u_n, u_{n+1} and their x-derivatives. Thence, any symmetry is equivalent to some coupled PDE system. It is a non-Abelian generalization of the Levi system (Levi, J. Phys. A 1981, Adler & Sokolov, arXiv: 2008.09174). The map $n \to n+1$ defines a Bäcklund transformation for this system.

For VL^1, the pair $(p,q) = (u_n, u_{n+1})$ satisfies, for any n, the system

$$\begin{cases} q_{t_2} = \ \ q_{xx} + 2q_x q + 2(qp)_x + 2[qp, q], \\ p_{t_2} = -p_{xx} + 2pp_x + 2(qp)_x + 2[qp, p]. \end{cases}$$

For VL^2, the pair $(p,q) = (u_n, u^{\mathrm{T}}_{n+1})$ satisfies

$$\begin{cases} q_{t_2} = \ \ q_{xx} + 2q_x q + 2(pq)_x + 2[pq, q], \\ p_{t_2} = -p_{xx} + 2p_x p + 2(qp)_x + 2[p, qp]. \end{cases}$$

Symmetries and constraints

In fact, there exists an infinite hierarchy of flows:

$$[D_{t_i}, D_{t_j}] = 0, \quad [D_{\tau_i}, D_{t_j}] = jD_{t_{j+i-1}},$$
$$[D_{\tau_i}, D_{\tau_j}] = (j-i)D_{\tau_{j+i-1}}, \qquad i,j \ge 1.$$

We only use symmetries that contain u_{n+k} with $|k| \le 2$.

Any linear combination of derivations

$$D_t = \mu_1(xD_{t_2} + D_{\tau_2}) + \mu_2(xD_x + D_{\tau_1}) + \mu_3 D_{t_2} + \mu_4 D_x$$

commute with D_x. Therefore, the stationary equation

$$D_t(u_n) = 0$$

is a constraint consistent with the lattice.

Up to equivalence transformations, there are three different cases which lead to (non-Abelian) Painlevé equations:

$$\begin{array}{llll} & 2(xD_x+D_{\tau_1}) \quad +D_{t_2} & =0 & \to \quad \mathrm{dP}_1+\mathrm{P}_4 \\ xD_{t_2}+D_{\tau_2} & +\mu(xD_x+D_{\tau_1}) & +\nu D_x=0 & \to \quad \mathrm{dP}_{34}+\mathrm{P}_5 \\ xD_{t_2}+D_{\tau_2} & & +\nu D_x=0 & \to \quad \mathrm{dP}_{34}+\mathrm{P}_3 \end{array}$$

- In all cases, we start from some 5-point OΔE

$$f_n(u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2};x,\mu,\nu)=0.$$

- It admits a reduction of order due to *partial* first integrals (pfi).
- The final result is a discrete Painlevé equation

$$g_n(u_{n-1},u_n,u_{n+1};x,\mu,\nu,\varepsilon,\delta)=0.$$

- It defines a subclass of special solutions of the original equation. Additional constants $\varepsilon,\delta\in\mathbb{C}$ replace two matrix initial data.
- The x-dynamics is also consistent with pfi. The VL is reduced to an ODE system for (u_n,u_{n+1}) which is equivalent to a continuous Painlevé equation.

Scaling reduction: $D_{t_2}+2(xD_x+D_{\tau_1})=0 \;\to\; \mathrm{dP}_1+\mathrm{P}_4$

$$\begin{aligned}\mathrm{VL}^1:\ &(u_{n+2}u_{n+1}+u^2_{n+1}+u_{n+1}u_n)u_n-u_n(u_nu_{n-1}+u^2_{n-1}+u_{n-1}u_{n-2})\\ &\quad+2x(u_{n+1}u_n-u_nu_{n-1})+2u_n=0,\end{aligned}$$

$$\begin{aligned}\mathrm{VL}^2:\ &(u^{\mathrm{T}}_{n+1}u_{n+2}+(u^{\mathrm{T}}_{n+1})^2+u_nu^{\mathrm{T}}_{n+1})u_n-u_n(u^{\mathrm{T}}_{n-1}u_n+(u^{\mathrm{T}}_{n-1})^2+u_{n-2}u^{\mathrm{T}}_{n-1})\\ &\quad+2x(u^{\mathrm{T}}_{n+1}u_n-u_nu^{\mathrm{T}}_{n-1})+2u_n=0.\end{aligned}$$

This can be represented as $F_{n+1}u_n-u_nF_{n-1}=0$.

The equality $F_n=0$ is pfi. Its consistency with D_x is due to identities:

- $F_{n,x}=(F_{n+1}-F_n)u_n+u_n(F_n-F_{n-1})$ for VL^1
- $F_{n,x}=(F^{\mathrm{T}}_{n+1}+F_n)u_n-u_n(F_n+F^{\mathrm{T}}_{n-1})$ for VL^2

Two analogs of dP_1

$$u_{n+1}u_n+u_n^2+u_nu_{n-1}+2xu_n+\gamma_n=0, \qquad \mathrm{dP}_1^1$$

$$u^{\mathrm{T}}_{n+1}u_n+u_n^2+u_nu^{\mathrm{T}}_{n-1}+2xu_n+\gamma_n=0, \qquad \mathrm{dP}_1^2$$

$$\gamma_n:=n-\nu+(-1)^n\varepsilon.$$

Continuous dynamics is as follows.

- VL^1, $(p,y)=(u_{n-1},u_n)$:

$$p_x=2yp+p^2+2xp+\gamma_{n-1},\quad y_x=-y^2-2yp-2xy-\gamma_n,$$

- VL^2, $(p,y)=(u^{\mathrm{T}}_{n-1},u_n)$:

$$p_x=2py+p^2+2xp+\gamma_{n-1},\quad y_x=-y^2-2yp-2xy-\gamma_n.$$

Two analogs of P_4

$$y''=\frac{1}{2}y'y^{-1}y'+[\kappa_i y-\gamma y^{-1},y']+\frac{3}{2}y^3+4xy^2+2(x^2-\alpha)y-2\gamma^2y^{-1},\quad \mathrm{P}^i_4$$

where $\alpha=\gamma_{n-1}-\gamma_n/2+1$, $\gamma=\gamma_n/2$,

$$\kappa_1=\frac{1}{2}\quad\text{and}\quad\kappa_2=-\frac{3}{2}.$$

- In the scalar case, this reduction was introduced by Its, Kitaev and Fokas [Russ. Math. Surv. 1990, Comm. Math. Phys. 1991].
- Another non-Abelian version of dP_1 was studied by Cassatella-Contra, Mañas and Tempesta [Stud. Appl. Math. 2012, Nonlinearity 2018]:

$$u_{n+1} + u_n + u_{n-1} + 2x + \gamma_n u_n^{-1} = 0.$$

Master-symmetry reduction:

$$xD_{t_2}+D_{\tau_2}+\mu(xD_x+D_{\tau_1})+\nu D_x=0 \quad\to\quad \mathrm{dP}_{34}+\mathrm{P}_5 \text{ or } \mathrm{P}_3$$

The first step is easy (like in the previous case). It brings to 4-point equations

$$\begin{aligned}\mathrm{VL}^1:\ & x(u_{n+2}u_{n+1}+u_{n+1}^2-u_n^2-u_nu_{n-1})-(2\mu x-n+\nu-\tfrac{3}{2})u_{n+1}\\ &\quad+(2\mu x-n+\nu+\tfrac{1}{2})u_n-\mu+2(-1)^n\varepsilon=0,\end{aligned}$$

$$\begin{aligned}\mathrm{VL}^2:\ & x\big(u^{\mathrm{T}}_{n+1}u_{n+2}+(u^{\mathrm{T}}_{n+1})^2-u_n^2-u_nu^{\mathrm{T}}_{n-1}\big)-(2\mu x-n+\nu-\tfrac{3}{2})u^{\mathrm{T}}_{n+1}\\ &\quad+(2\mu x-n+\nu+\tfrac{1}{2})u_n-\mu+2(-1)^n\varepsilon=0,\end{aligned}$$

where $\varepsilon\in\mathbb{C}$ is an integration constant. To obtain Painlevé equations, we need additional pfi.

In the scalar case, the above equation admits the integrating factor $xu_{n+1}+xu_n+n-\nu+\frac{1}{2}$ which brings to dP_{34}:

$$(z_{n+1}+z_n)(z_n+z_{n-1})=4x\frac{\mu z_n^2+2(-1)^n\varepsilon z_n+\delta}{z_n-n+\nu},\quad z_n:=2xu_n+n-\nu$$

[Adler & Shabat, Theor. Math. Phys. 2019].

Two analogs of dP_{34} for $\mu \neq 0$

$$(z_{n-1}+z_n)(z_n+(-1)^n\sigma+\omega)^{-1}(z_n+z_{n+1}) \\ = 4\mu x(z_n-n+\nu)^{-1}(z_n+(-1)^n\sigma-\omega), \qquad \mathrm{dP}^1_{34}$$

$$(z^{\mathrm{T}}_{n-1}+z_n)(z_n+(-1)^n(\sigma-\omega))^{-1}(z_n+z^{\mathrm{T}}_{n+1}) \\ = 4\mu x(z_n-n+\nu)^{-1}(z_n+(-1)^n(\sigma+\omega)) \qquad \mathrm{dP}^2_{34}$$

(where $\sigma=\varepsilon/\mu$, $\omega\in\mathbb{C}$).

Two analogs of dP_{34} for $\mu = 0$

$$\begin{cases} (z_{n+1}+z_n)(z_n-n+\nu)(z_n+z_{n-1}) = 4x(2\varepsilon z_n+\delta), & n=2k, \\ (z_n+z_{n-1})(z_{n+1}+z_n)(z_n-n+\nu) = 4x(-2\varepsilon z_n+\delta), & n=2k+1, \end{cases} \qquad \mathrm{d}\widetilde{\mathrm{P}}^1_{34}$$

$$(z^{\mathrm{T}}_{n+1}+z_n)(z_n-n+\nu)(z_n+z^{\mathrm{T}}_{n-1}) = 4x(2(-1)^n\varepsilon z_n+\delta). \qquad \mathrm{d}\widetilde{\mathrm{P}}^2_{34}$$

Equations dP^i_{34} and $\mathrm{d}\widetilde{\mathrm{P}}^i_{34}$ are consistent with VL^i. This gives rize to ODE systems for the variables $(q,p)=(z_n,z_n+z_{n+1})$ or $(z_n,z_n+z^{\mathrm{T}}_{n+1})$.

Two analogs of P_5

$$\mathrm{dP}^1_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4\mu x(q+\alpha)p^{-1}(q+\beta),\\ 2xp_x=pq+qp+p-p^2+4\mu x(p-2q-\alpha-\beta),\end{cases}\qquad \mathrm{P}^1_5$$

$$\mathrm{dP}^2_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4\mu x(q+\alpha)p^{-1}(q+\beta),\\ 2xp_x=2pq+p-p^2+4\mu x(p-2q-\alpha-\beta)\end{cases}\qquad \mathrm{P}^2_5$$

(in the scalar case, P_5 is satisfied by $y=1-4\mu xp^{-1}$).

Two analogs of P_3

$$\mathrm{d}\widetilde{\mathrm{P}}^1_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4xp^{-1}(2\varepsilon q+\delta),\\ 2xp_x=pq+qp+p-p^2-8\varepsilon x,\end{cases}\quad\text{(even } n)\qquad \mathrm{P}^1_3$$

$$\mathrm{d}\widetilde{\mathrm{P}}^2_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4xp^{-1}(2(-1)^n\varepsilon q+\delta),\\ 2xp_x=2pq+p-p^2-8(-1)^n\varepsilon x\end{cases}\qquad \mathrm{P}^2_3$$

(in the scalar case, P_3 is satisfied by $y=p/(2\xi)$, $x=\xi^2$).

Zero curvature representations

$$\mathrm{VL}^1: \quad u_{n,x} = u_{n+1}u_n - u_nu_{n-1} \quad \Leftrightarrow \quad L_{n,x} = U_{n+1}L_n - L_nU_n$$

$$L_n = \begin{pmatrix} \lambda & \lambda u_n \\ -1 & 0 \end{pmatrix}, \quad U_n = \begin{pmatrix} \lambda + u_n & \lambda u_n \\ -1 & u_{n-1} \end{pmatrix}$$

$$\mathrm{VL}^2: \quad u_{n,x} = u^{\mathrm{T}}_{n+1}u_n - u_nu^{\mathrm{T}}_{n-1} \quad \Leftrightarrow \quad L_{n,x} = U_{n+1}L_n + L_nU^{\mathrm{T}}_n$$

$$L_n = \begin{pmatrix} 1 & -\lambda \\ 0 & \lambda u_n \end{pmatrix}, \quad U_n = \begin{pmatrix} \frac{1}{2}\lambda & 1 \\ -\lambda u_{n-1} & -\frac{1}{2}\lambda - u_{n-1} + u^{\mathrm{T}}_n \end{pmatrix}$$

These are the compatiblity conditions, respectively, for

$$\Psi_{n+1} = L_n\Psi_n, \qquad \Psi_{n,x} = U_n\Psi_n$$

or for

$$\begin{aligned} \Psi_{2n+1} &= L_{2n}\Psi_{2n} & \Psi_{2n,x} &= -U^{\mathrm{T}}_{2n}\Psi_{2n}, \\ &= L^{\mathrm{T}}_{2n+1}\Psi_{2n+2}, & \Psi_{2n+1,x} &= U_{2n+1}\Psi_{2n+1}. \end{aligned}$$

More generally, any derivation from $\mathrm{VL}^1/\mathrm{VL}^2$ hierarchy admits a representation of the form

$$L_{n,t}+\kappa L_{n,\lambda}=V_{n+1}L_n-L_nV_n \qquad \text{or} \qquad L_{n,t}+\kappa L_{n,\lambda}=V_{n+1}L_n+L_nV_n^{\mathrm{T}},$$

with respective L_n and with certain V_n and $\kappa=\kappa(\lambda)$.

In both cases, we also have

$$U_{n,t}+\kappa U_{n,\lambda}=V_{n,x}+[V_n,U_n].$$

Therefore, for the stationary equation for D_t, we have the isomonodromic Lax pairs:

$$\kappa L_{n,\lambda}=V_{n+1}L_n-L_nV_n \qquad \text{or} \qquad \kappa L_{n,\lambda}=V_{n+1}L_n+L_nV_n^{\mathrm{T}}$$

for a discrete Painlevé equation and

$$\kappa U_{n,\lambda}=V_{n,x}+[V_n,U_n]$$

for a continuous one.

Explanation of dP^i_{34} partial first integral

Lemma. If $V_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies Lax equations

$$V_{n,x} = [U_n, V_n], \qquad V_{n+1}L_n = L_nV_n \quad \text{or} \quad V_{n+1}L_n = -L_nV_n^{\mathrm{T}}$$

then its quasi-determinant $\Delta_n = b - ac^{-1}d$ is pfi.

Proof. It is easy to derive relations of the form

$$\Delta_{n,x} = f\Delta - \Delta g, \qquad \Delta_{n+1} = f\Delta_n g \quad \text{or} \quad \Delta_{n+1} = f\Delta_n^{\mathrm{T}}g$$

which imply that the constraint $\Delta = 0$ is preserved.

The constraint $xD_{t_2} + D_{\tau_2} + \mu(xD_x + D_{\tau_1}) + \nu D_x = 0$ admits the isomonodromic Lax pairs with $\kappa(\lambda) = \lambda^2 - 2\mu\lambda$. For $\lambda = 2\mu$, the matrix $V_n - \alpha I$ satisfies Lax equations and vanishing of its quasi-determinant gives exactly dP^i_{34}.