Some exact solutions of the Volterra lattice

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We study solutions of the Volterra lattice satisfying the stationary equation for its non-autonomous symmetry. It is shown that the dynamics in t and n are governed by the continuous and discrete Painlevé equations, respectively. The class of initial data leading to regular solutions is described. For the lattice on the half-line, these solutions are expressed in terms of the confluent hypergeometric function. The Hankel transform of the coefficients of the corresponding Taylor series is computed on the basis of the Wronskian representation of the solution. arXiv:1903.11901

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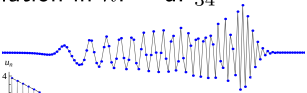
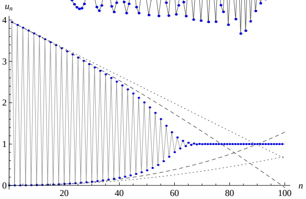
Plan of the talk

Volterra lattice (notations: $u_n = u_n(t)$, $u_{n,t} = \partial_t(u_n)$)

$$u_{n,t} = u_n(u_{n+1} - u_{n-1})$$

- General scheme: stationary equations for symmetries from the noncommutative Lie subalgebra $\Rightarrow$ Painlevé type reductions
- Master-symmetry + lower-order terms $\Rightarrow \begin{cases} \text{evolution in } t\text{: } \mathrm{P}_5 \text{ or } \mathrm{P}_3 \\ \text{evolution in } n\text{: } \mathrm{dP}_{34} \end{cases}$
- Regular solutions on the line ($n \in \mathbb{Z}$)
- Regular solutions on the half-line ($u_0 = 0$, $n > 0$)
    - Linearization:
      $\mathrm{P}_5 \Rightarrow$ confluent hypergeometric equation
    - Asymptotic expansions
    - Determinant identities





$$\begin{vmatrix} 1 & 1 & 2 & 5 & 14 \\ 1 & 2 & 5 & 14 & 42 \\ 2 & 5 & 14 & 42 & 132 \\ 5 & 14 & 42 & 132 & 429 \\ 14 & 42 & 132 & 429 & 1430 \end{vmatrix} = 1$$

General scheme

Let an evolution equation $u_t = f[u]$ admits a symmetry $u_\tau = g[t, u]$:

$$[\partial_t, \partial_\tau] = 0.$$

Then the stationary equation

$$g[t, u] = 0$$

is a constraint consistent with the dynamics in t.

If ∂_τ belongs to a commutative Lie subalgebra of the higher symmetries, then the stationary equation inherits this subalgebra. This is the case of finite-gap and multisoliton solutions, for instance.

In contrast, if ∂_τ contains some members of the additional noncommutative Lie subalgebra of symmetries then the constraint leads to a Painlevé type equation. This is the case which we are interested in.

Equations from the Volterra lattice hierarchy

We study solutions which satisfy the stationary equation for the master-symmetry + lower order terms. Here are all equations which we use.

The model itself:

$$u_{n,t} = u_n(u_{n+1} - u_{n-1}); \qquad \text{VL}$$

a higher symmetry:

$$u_{n,t_2} = u_n(h_{n+1} - h_{n-1}), \quad h_n := u_n(u_{n+1} + u_n + u_{n-1});$$

the scaling symmetry:

$$u_{n,\tau_0} = tu_n(u_{n+1} - u_{n-1}) + u_n;$$

the master-symmetry:

$$u_{n,\tau_1} = tu_n(h_{n+1} - h_{n-1}) + u_n\big((n + \tfrac{3}{2})u_{n+1} + u_n - (n - \tfrac{3}{2})u_{n-1}\big).$$

A simpler example

Let us use only the simplest noncommutative symmetry ∂_{τ_0}. This leads to equation

$$u_{n,t_2} + 2u_{n,\tau_0} = 0.$$

The coefficient at u_{n,τ_0} is fixed by scaling and the term $u_{n,t}$ can be omitted due to the shift $t \to t - \mathrm{const}$. This is a 5-point constraint (= involving $u_{n-2}, \dots, u_{n+2}$) which can be easily reduced to the 3-point one [Fokas, Its & Kitaev 1990, 1991]

$$u_n(u_{n+1} + u_n + u_{n-1}) + 2tu_n + n + (-1)^n b + c = 0 \qquad \mathrm{dP}_1$$

with integration constants b, c. This dP_1 equation turns the VL into a coupled system for u_{n-1}, u_n which is equivalent to P_4 equation for $y(t) = u_n(t)$:

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{2y}, \qquad \mathrm{P}_4$$

$$\alpha = \frac{1}{2}(n - 3(-1)^n b + c), \quad \beta = -(n + (-1)^n b + c)^2.$$

The map $(u_{n-1}, u_n) \mapsto (u_n, u_{n+1})$ is one of the Bäcklund transformations for P_4.

Our constraint

Let us consider more complicated case

$$u_{n,\tau_1} - 4au_{n,\tau_0} - du_{n,t} = 0.$$

Here, a can be scaled either to 0 or to 1 and the shift of t makes possible to omit the term u_{n,t_2}. Like in the FIK example, this is a 5-point constraint which can be reduced to a 3-point one, although it is not so obvious.

Statement 1. The VL is consistent, for any constants a, b, c, d, with the equation

$$F_n = (q_{n+1} + q_n)(q_n + q_{n-1})u_n - 4(aq_n^2 + (-1)^n bq_n + c) = 0, \tag{1}$$

where

$$q_n := 2tu_n + n - d.$$

Proof follows from the identity

$$F_{n,t} = u_n \frac{q_n + q_{n-1}}{q_{n+1} + q_n}(F_{n+1} - F_n) + u_n \frac{q_{n+1} + q_n}{q_n + q_{n-1}}(F_n - F_{n-1}).$$

The situation when $q_{n+1} + q_n = 0$ at some t requires an additional treatment, but one can show that it is not dangerous. □

We can completely rewrite equations in terms of q_n, then the constraint turns into the dP_{34} equation [Grammaticos, Ramani 2014]:

$$q_{n,t} = \frac{1}{2t}(q_n - n + d)(q_{n+1} - q_{n-1}),$$

$$(q_{n+1} + q_n)(q_n + q_{n-1}) = \frac{8t(aq_n^2 + (-1)^n b q_n + c)}{q_n - n + d}. \qquad \mathrm{dP}_{34}$$

Similar to the previous example, the VL turns into a coupled system for u_{n-1}, u_n, and the shift $n \mapsto n+1$ defines a Bäcklund transformation for the latter.

Statement 2. If $a \neq 0$ then functions

$$y_n(t) = 1 - \frac{8at}{q_{n+1}(t) + q_n(t)}$$

satisfy

$$y'' = \Big(\frac{1}{2y} + \frac{1}{y-1}\Big)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\Big(\alpha y + \frac{\beta}{y}\Big) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}, \qquad \mathrm{P}_5$$

$$\alpha = \frac{b^2 - 4ac}{8a^2}, \quad \beta = -\frac{(a + (-1)^n b)^2}{8a^2}, \quad \gamma = -2a(2n - 2d + 1), \quad \delta = -8a^2.$$

If $a=0$ then functions

$$y_n(z) = \frac{1}{2z}(q_{n+1}(t) + q_n(t)), \quad t = z^2,$$

satisfy

$$y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \qquad \mathsf{P}_3$$

$$\alpha = -4n + 4d - 2, \quad \beta = -4(-1)^n b - 8c, \quad \gamma = 4, \quad \delta = -16b^2.$$

Regular solutions on the line

In general, solutions of VL under constraint (1) have singularities in t. This is interesting, too, but here we will consider only a special family of solutions $u_n(t)$ which are continuously differentiable $\forall\ t \in \mathbb{R},\ n \in \mathbb{Z}$.

The regularity condition strictly fixes the initial data at $t = 0$. Indeed, since the values $u_n(0)$ are finite for a regular solution, hence $q_n(0) = n - d$ and we obtain directly from (1) that

$$u_n(0) = a + \frac{4(-1)^n b(n-d) + 4c + a}{4(n-d)^2 - 1} \qquad \text{if} \qquad d \notin \frac{1}{2} + \mathbb{Z}. \tag{2}$$

In the case when d is half-integer, this formula should be slightly changed:

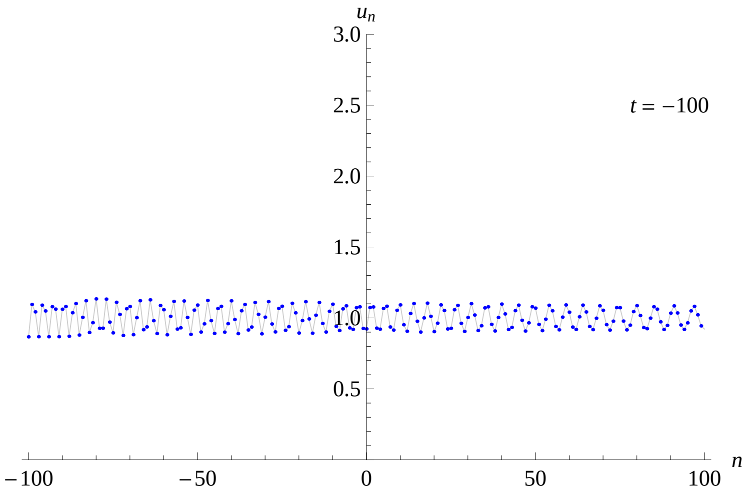
$$\begin{aligned} &u_n(0) = a + b\frac{(-1)^n(2n-2k-1) + (-1)^k}{2(n-k)(n-k-1)}, \quad n \neq k, k+1, \\ &u_{k+1}(0) + u_k(0) = 2a - 2(-1)^k b, \qquad d = \frac{1}{2} + k,\ k \in \mathbb{Z}. \end{aligned} \tag{3}$$

In any case, for the fixed values of a, b, c and d, we choose just one special solution from the 2-parametric family.

In this section, we will assume additionally that $u_n(0) \neq 0$ for all n.

Here is an example of such regular solution, corresponding to the choice

$$a = 1, \quad b = 0, \quad c = -3/4, \quad d = 0 \quad \Rightarrow \quad u_n(0) = 1 - \frac{2}{4n^2 - 1}.$$



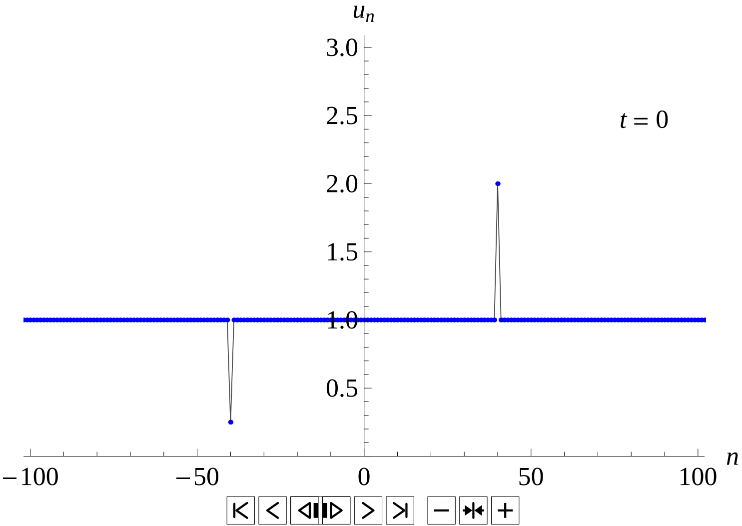
We see that initial data quickly collapse and generate an arrow-shaped zone of small-scale oscillations (with the period comparable to the lattice spacing) which expands at a constant speed in both directions.

For nonzero b and d, the initial data look a bit more complicated, but the general behavior of the solution remains the same.

Moreover, the picture does not change much if we take the initial data that do not satisfy the constraint (1), but are close to (2). Apparently, this behavior is typical for solutions with generic initial data in the form of sharp spikes, as opposed to solutions from the soliton sector which are formed when the initial data are relatively gently sloping (although there is no disctinct borderline between these two modes).

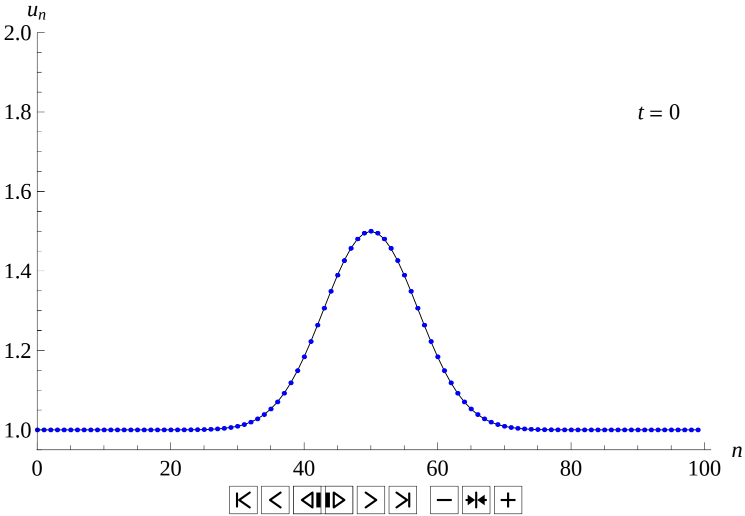
Thus, this is a fairly common mode in the Volterra lattice that deserves to be studied. It would be interesting to obtain its description from the point of view of the inverse scattering method. The constraint (1) is of interest as an exact solution example for this mode, in terms of the Painlevé transcendents.

A solution for initial data which differ from 1 at two points. Each spike generates oscillations of the type described which form an interference pattern after fusion.



For a comparison, here is a typical soliton-like solution, for the initial data

$$u_n(0) = 1 + 0.5\exp(-0.01(n-50)^2), \quad n = 0, \ldots, 99, \quad u_{n+100} = u_n.$$



Remark 1. The regularity at $t=0$ does not guarantee regularity for all t. The numeric experiments show that, for our solution family, the crucial property is related with the signs of $u_n(0)$: if all are the same then the solution is regular for all t; in contrast, if there are different signs then it acquires a singularity at a finite t.

If $a=0$ and $|b|+|c|\neq 0$ then $u_n(0)$ alternates and no regular solutions exist.

If $a=1$ and $d\in(-\frac{1}{2},\frac{1}{2}]$ (wlog) then the condition $u_n(0)>0$ reduces to

$$bd-c-d^2>0,\quad b(d-1)+c+(d-1)^2>0,\quad b(d+1)+c+(d+1)^2>0.$$

These inequalities cut off a bounded region in the parameter space and the corresponding solutions are regular.

Remark 2. The question about the regularity criterium for solutions with generic inital data is open. Regular solutions with different signs do exist: for instance,

$$u_{2n}=-\frac{\beta(n+\delta)e^{2\beta t}}{\alpha+e^{2\beta t}},\quad u_{2n+1}=\frac{\beta(\alpha n+\gamma)}{\alpha+e^{2\beta t}},\quad \alpha\geq 0$$

or the stationary solution $u_{2n}=\alpha$, $u_{2n+1}=\beta$ with constants of different signs. However, the non-alternating solutions are of primary interest. In many papers, this requirement is simply postulated; sometimes, the VL is wrtten down in the variables $p_n=\sqrt{u_n}$.

Regular solutions on the half-line

Under certain relations between a, b, c, d, it is possible that the equality $u_m(0) = 0$ is fulfilled which splits the lattice (VL) into two independent systems for $n < m$ and $n > m$. Up to the shift of n, it is sufficient to consider the case of half-line

$$u_0 = 0, \quad u_n \neq 0, \quad n > 0.$$

The ODE system for u_0, u_1 (equivalent, in general, to P_5) reduces to the Riccati equation for u_1.

Statement 3. The function $u_1(t)$ satisfies the equation

$$u_1' + u_1^2 - \Big(4a + \frac{2d-3}{2t}\Big)u_1 - \frac{2(a-b)}{t} = 0 \tag{4}$$

which is linearized by the substitution $u_1 = f'/f$:

$$tf'' + \Big(\frac{3}{2} - d - 4at\Big)f' - 2(a-b)f = 0. \tag{5}$$

As before, we will consider only regular solutions. The corresponding initial data are

$$u_1(0) = \frac{4(b-a)}{2d-3}, \quad u_n(0) = \frac{a(n-d)^2 + (-1)^n b(n-d) + d(b-ad)}{(n-d)^2 - 1/4}, \quad n > 1, \quad (6)$$

with some restrictions which guarantee that $u_n \neq 0$ for $n > 0$:

$$d \neq \frac{1}{2} + k, \quad b \neq a(2k-1), \quad b \neq 2a(d-k), \quad k = 1, 2, 3, \dots \quad (7)$$

- **Case $a = 0$**

If $b \neq 0$ then the change $z = 2\sqrt{2bt}$, $f(t) = t^{d/2-1/4} y(z)$ brings (5) to the Bessel equation

$$z^2 y'' + z y' + \big(z^2 - (d - \tfrac{1}{2})^2\big) y = 0.$$

The initial data are alternating. Numeric experiments show that if $b < 0$ (wlog) then the solution acquires the poles at $t < 0$, but it tends to 0 for $t > 0$. This gives an example of alternating solution which is bounded and regular in the quadrant $n, t > 0$. However, it is very unstable with respect to the calculation errors and the perturbations of the initial data.

Case $a \neq 0$

This is more interesting case. The change $z = 4at$, $f(t) = y(z)$ brings (5) to the confluent hypergeometric equation

$$zy'' + (\beta - z)y' - \alpha y = 0, \quad \alpha = \frac{a-b}{2a}, \quad \beta = \frac{3}{2} - d.$$

Regularity of $u_1 = f'/f$ at $t = 0$ means that we choose the Kummer function as the solution:

$$f(t) = M(\alpha, \beta, 4at).$$

In the rest of the talk we will discuss:

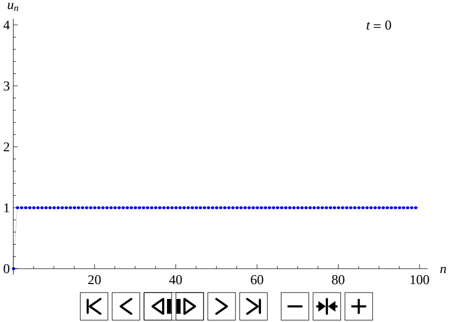
- general behaviour of solutions;
- asymptotic expansions;
- determinant identities for the Taylor coefficients of $f(t)$.

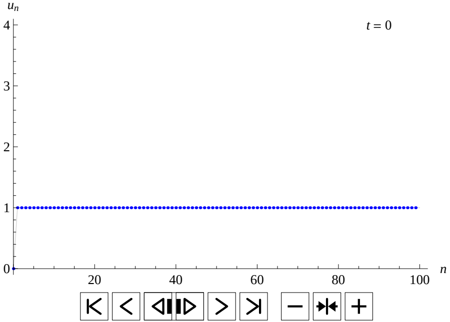
Decay of step-like initial data

If $a = 1$ (wlog), $b = 0$ and $d = -\frac{1}{2}$ then the initial data take especially simple form of the unit step:

$$u_n(0) = 1 \quad \text{for} \quad n > 0.$$

The corresponding solution was constructed in [Adler & Shabat 2018].



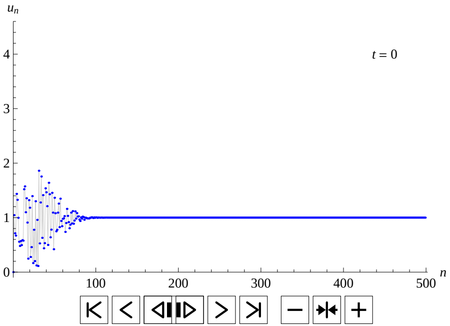


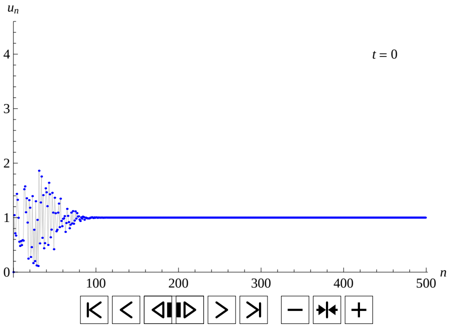
The dashed lines correspond to one or two terms of the asymptotic expansions.

For other admissible values of b and d, the general behaviour remains the same. Numeric experiments demonstrate that this mode is stable with respect to small enough perturbations of initial data (of course, preserving the boundary value $u_0 = 0$). Here is the evolution of the initial data

$$u_n(0) = 1 + 0.9\exp(-0.001(n-30)^2)r_n, \quad n > 0,$$

where r_n is a random value uniformly distributed in $[-1, 1]$.





This perturbation results in soliton-like structures on the background with the same asymptotics.

Asymptotic expansions

Let $a = 1$, without loss of generality, so that $z = 4t$.

In order to obtain the asymptotic, we compare the known formulas

$$M(\alpha, \beta, z) = \begin{cases} \dfrac{\Gamma(\beta)}{\Gamma(\alpha)} e^z z^{\alpha-\beta}\big(1 + O(|z|^{-1})\big), & \operatorname{Re} z > 0, \\ \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (-z)^{-\alpha}\big(1 + O(|z|^{-1})\big), & \operatorname{Re} z < 0 \end{cases}$$

with the formal expansion

$$u_1 = q_0 + q_1 t^{-1} + q_2 t^{-2} + \dots$$

Substitution into the Riccati equation (4) gives

$$q_0 = 4 \quad \text{or} \quad q_0 = 0$$

and all subsequent coefficients are computed uniquely. Therefore,

$$u_1 \sim \begin{cases} 4 - (\alpha - \beta) t^{-1} + \dots, & t \to +\infty, \\ 0 + \alpha t^{-1} + \dots, & t \to -\infty. \end{cases}$$

Expansions for all u_n are obtained by direct substitution into the VL equations.

Statement 4. For $t \to +\infty$

$$u_n = \begin{cases} \dfrac{n(n-2d+b)}{16t^2} + \dfrac{n(n-2d+b)(2n-2d+3b)}{128t^3} + O(t^{-4}), & n = 0, 2, \dots, \\ 4 - \dfrac{2n-2d+b}{2t} \\ \qquad - \dfrac{n^2-(2d-3b)n+b^2-2bd+1}{16t^2} + O(t^{-3}), & n = 1, 3, \dots \end{cases}$$

and for $t \to -\infty$

$$u_n = \begin{cases} -\dfrac{n}{2t} + \dfrac{(2d-b)n}{16t^2} + O(t^{-3}), & n = 0, 2, \dots, \\ -\dfrac{n-b}{2t} + \dfrac{(2d-b)(n-b)}{16t^2} + O(t^{-3}), & n = 1, 3, \dots \end{cases}$$

▪ A remark on the conservation laws

The VL admits an infinite sequence of conservation laws

$$\frac{d}{dt}\rho_n^{(k)} = \sigma_{n+1}^{(k)} - \sigma_n^{(k)},$$

three simplest ones are

$$\rho_n^{(0)} = \log u_n,\ \sigma_n^{(0)} = u_{n-1} + u_n, \qquad \rho_n^{(1)} = u_n,\ \sigma_n^{(1)} = u_{n-1}u_n,$$
$$\rho_n^{(2)} = \frac{1}{2}u_n^2 + u_n u_{n+1},\ \sigma_n^{(2)} = u_{n-1}u_n(u_n + u_{n+1}).$$

If $n \in \mathbb{Z}$ and $u_{-\infty} = u_{+\infty}$ then the quantities

$$H_k = \sum_n (\rho_n^{(k)} - r^{(k)})$$

are preserved, where $r^{(k)}$ is a suitable normalizing constant.

In contrast, if $u_0 = 0$ then the analogous sums over $n > 0$ do not preserve, since

$$\frac{d}{dt}H_k = \sigma_{+\infty}^{(k)} - \sigma_1^{(k)} \neq 0.$$

In particular, if $u_{+\infty} = 1$ then the sums are regularized as follows:

$$H_0 = \sum_{n=1}^{\infty} \log u_n, \quad H_1 = \sum_{n=1}^{\infty} (u_n - 1), \quad H_2 = \sum_{n=1}^{\infty} \Big(\frac{1}{2} u_n^2 + u_n u_{n+1} - \frac{3}{2} \Big),$$

and we have

$$\sigma_{+\infty}^{(0)} = 2,\ \sigma_1^{(0)} = u_1; \quad \sigma_{+\infty}^{(1)} = 1,\ \sigma_1^{(1)} = 0; \quad \sigma_{+\infty}^{(2)} = 2,\ \sigma_1^{(2)} = 0.$$

Then

$$\frac{d}{dt} H_0 = 2 - u_1, \quad \frac{d}{dt} H_1 = 1, \quad \frac{d}{dt} H_2 = 2,$$

and since all three sums are equal to 0 at $t = 0$, hence

$$H_0 = \int_0^t (2 - u_1(\tau)) d\tau, \quad H_1 = t, \quad H_2 = 2t.$$

▪ Determinant identities

General solution on the half-line admits the following Wronskian representation.

Statement 5. [Leznov 1980] Let $u_0 = 0$ and $u_1 = f'/f$ with an arbitrary infinitely differentiable function $f(t)$. Then the solution of the VL for $n \geq 0$ is of the form

$$u_n = \frac{w_{n-3}w_n}{w_{n-2}w_{n-1}}, \quad n = 0, 1, 2, \dots,$$

where $w_{-3} = 0$, $w_{-2} = w_{-1} = 1$ and, for $k \geq 0$,

$$w_{2k} = \begin{vmatrix} f & f' & \dots & f^{(k)} \\ f' & f'' & \dots & f^{(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(k)} & f^{(k+1)} & \dots & f^{(2k)} \end{vmatrix}, \quad w_{2k+1} = \begin{vmatrix} f' & f'' & \dots & f^{(k+1)} \\ f'' & f''' & \dots & f^{(k+2)} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(k+1)} & f^{(k+2)} & \dots & f^{(2k+1)} \end{vmatrix}.$$

Although these expressions are not very convenient for a practical computing of solutions, we will show that using it together with explicit expressions for $u_1(t)$ and $u_n(0)$ makes possible to obtain nontrivial identities for some number sequences.

As an immediate corollary, we obtain the following relation between the Taylor coefficients of $f(t)$ and initial data $u_n(0)$.

Statement 6. Let a solution $u_n(t)$ of VL on the half-line be given by above Wronskian formulas with

$$f(t) = f_0 + f_1 t + \cdots + f_n \frac{t^n}{n!} + \cdots$$

Then, for $k = 0, 1, 2, \ldots,$

$$h_{2k} = \begin{vmatrix} f_0 & \ldots & f_k \\ \vdots & \ddots & \vdots \\ f_k & \ldots & f_{2k} \end{vmatrix} = \prod_{j=1}^{k} (u_{2j-1}(0)u_{2j}(0))^{k+1-j},$$

$$h_{2k+1} = \begin{vmatrix} f_1 & \ldots & f_{k+1} \\ \vdots & \ddots & \vdots \\ f_{k+1} & \ldots & f_{2k+1} \end{vmatrix} = u_1^{k+1}(0) \prod_{j=1}^{k} (u_{2j}(0)u_{2j+1}(0))^{k+1-j}.$$

Proof. By setting $h_n = w_n(0)$, we obtain $h_{-2} = h_{-1} = h_0 = 1$ and the recurrent relation

$$h_n h_{n-3} = u_n(0) h_{n-1} h_{n-2}, \quad n = 1, 2, \ldots,$$

which proves the statement by induction. □

The determinants of such form are actively studied in combinatorics. The sequence $h_0, h_2, h_4, \ldots$ is the so-called Hankel transformation for the sequence $f_0, f_1, f_2, \ldots$ and $h_1, h_3, h_5, \ldots$ is the Hankel transformation for $f_1, f_2, f_3, \ldots$

Example 1. Consider the explicit solution

$$u_{2k-1} = e^t, \quad u_{2k} = k.$$

Then

$$f'/f = u_1 = e^t \quad \Rightarrow \quad f(t) = e^{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

(the integration constant is not essential), where B_n are the Bell numbers

$$1,\ 1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 203,\ 877,\ 4140, \ldots$$

enumerating the partitions of a set of n elements. Since

$$u_{2k-1}(0) = 1, \quad u_{2k}(0) = k,$$

we obtain easily that $h_{2k} = h_{2k+1} = 1! \cdots k!$ (the superfactorial of k). This result is known since 1978 [Ehrenborg 2000].

Now let us apply Statement 6 to solutions from our family, with the inital data (6).

Due to the known expansion of the Kummer function, we have

$$f(t) = M(\alpha, \beta, 4at) = 1 + 4a\frac{\alpha}{\beta}t + (4a)^2\frac{(\alpha)_2}{(\beta)_2}\frac{t^2}{2!} + \cdots + (4a)^n\frac{(\alpha)_n}{(\beta)_n}\frac{t^n}{n!} + \dots,$$

where $(\alpha)_n$ is the Pochhammer symbol

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \quad (\alpha)_0 = 1.$$

Example 2. Let $a = 1$, $b = 0$ and $d = -\frac{1}{2}$ (or, $\alpha = \frac{1}{2}$ and $\beta = 2$). Then

$$u_n(0) = 1, \quad n > 0 \quad \text{and} \quad f_n = \frac{(2n)!}{(n+1)!n!},$$

that is, f_n are the Catalan numbers

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430, \ldots$$

The recurrence $h_n h_{n-3} = u_n(0) h_{n-1} h_{n-2}$ implies immediately that all $h_n = 1$, a well known property of the Catalan numbers [Aigner 1999, Stanley 1999, Layman 2001].

Example 3. Let $a = 1$, $b = 0$ and $d = \frac{1}{2}$ (or, $\alpha = \frac{1}{2}$ and $\beta = 1$). Then

$$u_1(0) = 2, \quad u_n(0) = 1, \quad n > 1 \quad \text{and} \quad f_n = \frac{(2n)!}{(n!)^2}$$

(the central binomial coefficients). Applying the recurrence relation proves then $h_{2k} = 2^k$, $h_{2k+1} = 2^{k+1}$. This example is also known in the combinatorics.

For the general initial data (6), we obtain the following result.

Statement 7. Let

$$f_n = (4a)^n \frac{(\alpha)_n}{(\beta)_n}, \quad n = 0, 1, 2, \dots,$$

$$\alpha \neq -n, \quad \beta \neq -n, \quad \alpha - \beta \neq n, \quad n = 0, 1, 2, \dots,$$

then, for $k = 0, 1, 2, \dots,$

$$h_{2k} = \begin{vmatrix} f_0 & \dots & f_k \\ \vdots & \ddots & \vdots \\ f_k & \dots & f_{2k} \end{vmatrix} = \frac{((1))_k (4a)^{k(k+1)} ((\alpha))_k ((\beta - \alpha))_k}{(\beta)_k^{k+1} ((\beta + k))_k},$$

$$h_{2k+1} = \begin{vmatrix} f_1 & \dots & f_{k+1} \\ \vdots & \ddots & \vdots \\ f_{k+1} & \dots & f_{2k+1} \end{vmatrix} = \frac{((1))_k (4a)^{(k+1)^2} ((\alpha))_{k+1} ((\beta - \alpha))_k}{(\beta)_k^{k+1} ((\beta + k))_{k+1}},$$

where

$$((\alpha))_n = (\alpha)_1 \cdots (\alpha)_n = \alpha^n (\alpha + 1)^{n-1} \cdots (\alpha + n - 1)^1, \quad ((\alpha))_0 = 1,$$

in particular, the superfactorial is denoted as $((1))_n = 1! \cdots n!$.