On the combinatorics of several integrable hierarchies

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We demonstrate that statistics of certain classes of set partitions is described by generating functions related to the Burgers, Ibragimov–Shabat and Korteweg–de Vries integrable hierarchies.

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Introduction

The inverse problem of enumerative combinatorics is formulated as search of objects with prescribed statistics. Let us look at integrable equations from this particular point of view. For instance, what is the combinatorics behind the KdV mysterious coefficients? Is it possible to figure out a definition of corresponding combinatorial objects?

$$
\begin{aligned}
u_{t_1} &= u_1 \\
u_{t_3} &= u_3 + 6uu_1 \\
u_{t_5} &= u_5 + (10uu_3 + 20u_1u_2) + 30u^2u_1 \\
u_{t_7} &= u_7 + (14uu_5 + 42u_1u_4 + 70u_2u_3) \\
&\qquad + (70u^2u_3 + 280uu_1u_2 + 70u_1^3) + 140u^3u_1 \\
u_{t_9} &= u_9 + (18uu_7 + 72u_1u_6 + 168u_2u_5 + 252u_3u_4) \\
&\qquad + (126u^2u_5 + 756uu_1u_4 + 1260uu_2u_3 + 966u_1^2u_3 + 1302u_1u_2^2) \\
&\qquad + (420u^3u_3 + 2520u^2u_1u_2 + 1260uu_1^3) + 630u^4u_1
\end{aligned}
$$

Korteweg–de Vries hierarchy, $w(u_j) = j + 2$

Contents of the talk

Hierarchy		Combinatorial objects, their numbers
potential Burgers	$=$	Set partitions, Bell polynomials Y_n, Stirling numbers of the 2nd kind, Bell numbers
Burgers	$=$	Set partitions without distinguished singleton
Ibragimov–Shabat	$\approx$	B-type partitions, B-analogs of Stirling numbers of the 2nd kind, Dowling numbers
KdV	$\approx$	Non-overlapping partitions, Bessel numbers

new results

Both left and right parts of the table are very well studied, but their relation deserves better understanding.

The combinatorial interpretation of equations was pointed out only in the simplest case of the pot-Burgers hierarchy, see e.g. Lambert et al (1994). The combinatorics related with solutions is better studied (τ-functions expansions and Hurwitz numbers, solitons and Bernoulli numbers, self-similar solutions and asymptotic of longest increasing subsequence of a random permutation, and more), but this is beyond our talk.

What is computed?

— For the pot-Burgers and Burgers hierarchies, we consider generating function intermediately for the higher flows.

— For the KdV, we compute the formal expansion of the log-derivative of ψ-function by solving the Riccati equation (inversion of the Miura map, see e.g. Gelfand & Dikii 1975). The KdV flows are related with this series by simple algebraic equations.

— In the IS case a natural choice of generating function is dictated by the linearization procedure.

How to compute?

For our purpose, it is enough to use recurrent relations.

We are not interested in "explicit" expressions for the coefficients here. However, it should be mentioned that such formulae for the pot-KdV flows actually do exist. One of them, obtained already by GD (1975) represents the coefficient of a given monomial as a certain multiple integral. Another expression (Schimming 1995) is "more combinatorial", but it remains very complicated. Only the pot-Burgers coefficients are given explicitly, indeed.

Potential Burgers hierarchy

It appears from the linear heat equation hierarchy

$$\psi_{t_n} = \psi_n, \quad n = 0, 1, 2, \ldots$$

after the change of dependent variable $\psi = e^v$:

$$v_{t_n} = e^{-v} D^n(e^v) = (D + v_1)^n(1) = Y_n(v_1, \ldots, v_n). \tag{1}$$

A meaningful combinatorics appears just from nothing!

$$
\begin{aligned}
v_{t_0} &= 1 \\
v_{t_1} &= v_1 \\
v_{t_2} &= v_2 + v_1^2 \\
v_{t_3} &= v_3 + 3v_1v_2 + v_1^3 \\
v_{t_4} &= v_4 + (4v_1v_3 + 3v_2^2) + 6v_1^2v_2 + v_1^4 \\
v_{t_5} &= v_5 + (5v_1v_4 + 10v_2v_3) + (10v_1^2v_3 + 15v_1v_2^2) + 10v_1^3v_2 + v_1^5
\end{aligned}
$$

The pot-Burgers hierarchy, $w(v_j) = j$

Polynomials Y_n play a fundamental role in many sciences and are known under the name of (full exponential) Bell polynomials, see e.g. Comtet (1974). An equivalent definition through the exponential generating functions reads

$$\sum_{n=0}^{\infty} Y_n \frac{z^n}{n!} = e^{-v} \sum_{n=0}^{\infty} D^n(e^v) \frac{z^n}{n!} = e^{v(x+z)-v(x)} = \exp\left(\sum_{n=1}^{\infty} v_n \frac{z^n}{n!}\right),$$

and this immediately implies the explicit formula

$$Y_n = \sum_{k_1+2k_2+\cdots+rk_r=n} \frac{n!}{(1!)^{k_1}\ldots(r!)^{k_r} k_1!\ldots k_r!} v_1^{k_1}\ldots v_r^{k_r}. \tag{2}$$

Its combinatorial interpretation is obvious:

— monomials correspond to partitions of the number n;

— coefficients of monomials count partitions of the set $[n] = \{1,\ldots,n\}$ into the subsets (or blocks) of prescribed size. Recall, that a set partition is considered as unordered set (with blocks as the elements), that is, ordering of the blocks does not matter.

Theorem 1. In the pot-Burgers hierarchy, the coefficient of $v_1^{k_1} \dots v_r^{k_r}$ is equal to the number of partitions with $n = k_1 + 2k_2 + \cdots + rk_r$ elements into k_1 blocks with 1 element, ..., k_r blocks with r elements.

$n=2:$	v_2	v_1^2
	2	$1+1$
	12	$1\|2$

$n=3:$	v_3	$3v_1v_2$	v_1^3
	3	$1+2$	$1+1+1$
	123	$1\|23$	$1\|2\|3$
		$2\|13$	
		$3\|12$	

$n=4:$	v_4	$4v_1v_3$	$3v_2^2$	$6v_1^2v_2$	v_1^4
	4	$1+3$	$2+2$	$1+1+2$	$1+1+1+1$
	1234	$1\|234$	$12\|34$	$1\|2\|34$	$1\|2\|3\|4$
		$2\|134$	$13\|24$	$1\|3\|24$	
		$3\|124$	$14\|23$	$1\|4\|23$	
		$4\|123$		$2\|3\|14$	
				$2\|4\|13$	
				$3\|4\|12$	

A less detailed statistics is obtained if we forget about sizes of blocks and consider just their number in a given partition. Obviously, this correspond to summing up the coefficients of terms of the same degree, which gives us the Bell polynomials of one variable

$$B_n(u) = Y_n(u, \dots, u) = (u\partial_u + u)^n(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} u^k.$$

The coefficient $\left\{ {n \atop k} \right\}$ of u^k is the number of partitions of $[n]$ into k blocks. It is called the Stirling number of the second kind (OEIS:A048993):

1							**1**
0	1						**1**
0	1	1					**2**
0	1	3	1				**5**
0	1	7	6	1			**15**
0	1	15	25	10	1		**52**
0	1	31	90	65	15	1	**203**

The sums over the rows, the Bell numbers $B_n = B_n(1) = Y_n(1, \dots, 1)$ give the total number of partitions (OEIS:A000110).

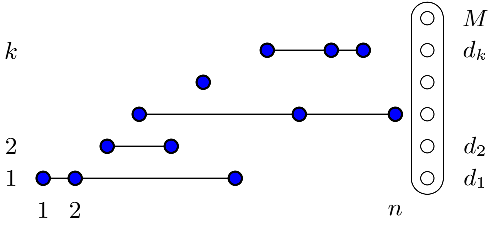
Proof of Theorem 1.

One proof follows intermediately from the explicit formula (2) for the coefficients. However, we will not always have such a formula at hand. The following reasoning is more conceptual.

Let $\Pi_{n,k}$ denote the set of all partitions of the set $[n]$ into k blocks and Π_n denote the set of all partitions of $[n]$. Consider the operations

$$d_j : \Pi_{n,k} \to \Pi_{n+1,k}, \quad j = 1, \ldots, k, \qquad M : \Pi_{n,k} \to \Pi_{n+1,k+1},$$

defined, respectively, as appending of the element $n+1$ to j-th block (we will define the enumeration of the blocks by ordering of their minimal elements) or as a new singleton:



Starting from the partition $\{\varnothing\}$ of the set $[0] = \varnothing$ and applying operations d_j, M, one can generate any partition of $[n]$, in a unique way. Indeed, the required sequence of operations is uniquely recovered by deleting elements in the inverse order from n to 1.

Recall that in the theorem, a set partition π with k_1 1-blocks, ..., k_r r-blocks corresponds to the monomial $p(\pi) = v_1^{k_1} \dots v_r^{k_r}$.

The differentiation $D(p(\pi))$ by the Leibnitz rule amounts to replacing of v_i with v_{i+1} for each factor in turn, taking the multiplicity into account. In the partition language, this means that we add the new element to each block in turn. As the result, we obtain the sum of monomials $p(d_j(\pi))$ for all admissible values of j.

Multiplication of $p(\pi)$ by v_1 gives the monomial $p(M(\pi))$.

Thus, the polynomials

$$P_n = \sum_{\pi \in \Pi_n} p(\pi)$$

are related by recurrent relation $P_{n+1} = (D + v_1)(P_n)$ and since $P_1 = v_1$, hence $P_n = Y_n(v_1, \dots, v_n)$. ∎

Burgers hierarchy

The right hand sides of equations (1) do not contain v and this makes the substitution $u = v_1$ possible. This brings to the Burgers hierarchy

$$u_{t_n} = D(Y_n(u, \ldots, u_{n-1})), \quad n = 1, 2, \ldots \,.$$

What is the combinatorial interpretation in this case?

$$\begin{aligned}
u_{t_1} &= u_1 \\
u_{t_2} &= u_2 + 2uu_1 \\
u_{t_3} &= u_3 + (3uu_2 + 3u_1^2) + 3u^2u_1 \\
u_{t_4} &= u_4 + (4uu_3 + 10u_1u_2) + (6u^2u_2 + 12uu_1^2) + 4u^3u_1 \\
u_{t_5} &= u_5 + (5uu_4 + 15u_1u_3 + 10u_2^2) + (10u^2u_3 + 50uu_1u_2 + 15u_1^3) \\
&\quad + (10u^3u_2 + 30u^2u_1^2) + 5u^4u_1
\end{aligned}$$

Burgers hierarchy, $w(u_j) = j + 1$

This can be easily understood by the following example, for $n = 3$:

$$
\begin{array}{ccc}
v_3 & 3v_1v_2 & v_1^3 \\
u_2 & 3uu_1 & u^3 \\
123 & 1|23 & 1|2|3 \\
 & 2|13 & \\
 & 3|12 &
\end{array}
\quad \xrightarrow{D} \quad
\begin{array}{ccccc}
u_3 & \multicolumn{2}{c}{3uu_2 + 3u_1^2} & 3u^2u_1 \\
1234 & 1|234 & 14|23 & 1|2|34 \\
 & 2|134 & 24|13 & 1|24|3 \\
 & 3|124 & 34|12 & 14|2|3
\end{array}
$$

$$\nearrow$$

$$
\begin{array}{ccccc}
u_3 & 3uu_2 & 3u_1^2 & 3u^2u_1 & u^4 \\
1234 & 1|234 & 12|34 & 1|2|34 & \cancel{1|2|3|4} \\
 & 2|134 & 13|24 & 1|3|24 & \\
 & 3|124 & 14|23 & \cancel{1|4|23} & \\
 & \cancel{4|123} & & 2|3|14 & \\
 & & & \cancel{2|4|13} & \\
 & & & \cancel{3|4|12} &
\end{array}
$$

Certainly, renaming $v_j \to u_{j-1}$ does not change the combinatorics.

The differentiation amounts to appending the new element to all blocks in turn, however, now we do not add it as a new block. Therefore, the partitions under consideration are constructed as before, but we do not apply the operation M at the last step.

As a result, all partitions Π_n are mapped onto those partitions from Π_{n+1} where the element $n+1$ does not appear as a singleton.

Theorem 2. In the Burgers hierarchy, the coefficient of $u^{k_0} u_1^{k_1} \dots u_r^{k_r}$ is equal to the number of partitions of the set with one distinguished element into k_0 blocks with 1 element, …, k_r blocks with $(r+1)$ element and such that the distinguished element does not constitute 1-block.

As before, one can consider more rough statistics. For instance, setting $u = 1$ gives us the total number of partitions under consideration of the set $[n+1]$:

$$D(Y_n(u, \dots, u_{n-1}))|_{u_j=1} = B'_n(1) = \sum_{k=1}^{n} k \begin{Bmatrix} n \\ k \end{Bmatrix}, \quad n \geq 1.$$

This integer sequence (2-Bell numbers) starts

$$1,\ 3,\ 10,\ 37,\ 151,\ 674,\ 3263,\ 17007,\ 94828,\ 562595,\ \ldots$$

According to (OEIS:A005493), it can be characterized also in many other ways, in particular, as the number of partitions of $[n]$ with distinguished block or as the total number of blocks in all set partitions of $[n]$. These interpretations are obvious as well, since the distinguished blocks can be identified with the blocks enlarged by the operations d_j, and these operations are applied exactly as many times as there are blocks in all partitions.

Ibragimov–Shabat hierarchy

Recurrent relations

Let us recall the sequence of point changes and substitutions between equation $\psi_{t_3} = \psi_3$ and the Ibragimov–Shabat equation (1980)

$$u_{t_3} = u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1.$$

$$\begin{array}{ccc}
\psi_{t_3} = \psi_3 & & u_{t_3} = u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1 \\
\updownarrow \psi^2 = s & & \updownarrow u^2 = v \\
s_{t_3} = D\left(s_2 - \dfrac{3s_1^2}{4s}\right) & & v_{t_3} = D\left(v_2 - \dfrac{3v_1^2}{4v} + 3vv_1 + v^3\right) \\
\uparrow s = q_1 & & \uparrow v = w_1 \\
q_{t_3} = q_3 - \dfrac{3q_2^2}{4q_1} & \xleftrightarrow{q = e^{2w}} & w_{t_3} = w_3 - \dfrac{3w_2^2}{4w_1} + 3w_1 w_2 + w_1^3
\end{array}$$

Linearization of the IS equation

This transformation spoils the even flows $\psi_{t_{2m}} = \psi_{2m}$. Indeed, the change $\psi^2 = s$ brings to equation $s_{t_n} = \dots \in \operatorname{Im} D$ only for odd n:

$$s_{t_n} = 2\psi\psi_n = D(2\psi\psi_{n-1} - 2\psi_1\psi_{n-2} + 2\psi_2\psi_{n-3} + \dots \pm \psi^2_{(n-1)/2}). \quad (3)$$

In the analogous equation for even n, the term $\psi^2_{n/2}$ remains outside the parentheses, that is $s_{t_n} \notin \operatorname{Im} D$, and the further substitution $s = q_1$ leads out of the class of evolutionary equations.

Statement. The IS hierarchy is equivalent to equations

$$D_t(u) = \frac{1}{2u} D(A\bar{A}) = \frac{1}{2z}(A - \bar{A}) - uA\bar{A}, \quad (4)$$

$$z(D + u^2)(A) = A - u \quad (5)$$

where $A = A(z)$, $\bar{A} = A(-z)$,

$$D_t = \partial_{t_1} + z^2\partial_{t_3} + z^4\partial_{t_5} + \dots, \quad A = a_0 + a_1 z + a_2 z^2 + \dots$$

Proof. Let us consider the generating function

$$\Psi = \psi + \psi_1 z + \psi_2 z^2 + \dots$$

and set $\Psi = \sqrt{2} e^w A$. Equation (5) follows from

$$zD(\Psi) = \Psi - \psi, \quad \psi = \sqrt{q_1} = \sqrt{2e^{2w} w_1} = \sqrt{2} e^w u.$$

Next, let $\bar{\Psi} = \Psi(-z)$, then (cf (3))

$$\begin{aligned} D(\Psi\bar{\Psi}) &= z^{-1}(\Psi - \psi)\bar{\Psi} - z^{-1}\Psi(\bar{\Psi} - \psi) \\ &= z^{-1}\psi(\Psi - \bar{\Psi}) = 2\psi(\psi_1 + \psi_3 z^2 + \dots) = 2\psi D_t(\psi) = D_t(s). \end{aligned}$$

Applying D^{-1} yields $\Psi\bar{\Psi} = D_t(q) = 2e^{2w} D_t(w)$, wherefrom

$$2uD_t(u) = D_t(v) = DD_t(w) = \frac{1}{2} D(e^{-2w}\Psi\bar{\Psi}) = D(A\bar{A}).$$

Second equality in (4) follows after elimination of derivatives by use of (5). ■

Equation (5) is equivalent to recurrent relations

$$a_0 = u, \quad a_n = a_n(u, \ldots, u_n) = (D + u^2)(a_{n-1}), \quad n = 1, 2, \ldots \tag{6}$$

which are our object of study. Let us try to find a combinatorial interpretation for this recursion.

$$\begin{aligned}
a_0 &= u \\
a_1 &= u_1 + u^3 \\
a_2 &= u_2 + 4u^2u_1 + u^5 \\
a_3 &= u_3 + (5u^2u_2 + 8uu_1^2) + 9u^4u_1 + u^7 \\
a_4 &= u_4 + (6u^2u_3 + 26uu_1u_2 + 8u_1^3) + (14u^4u_2 + 44u^3u_1^2) + 16u^6u_1 + u^9 \\
a_5 &= u_5 + (7u^2u_4 + 38uu_1u_3 + 26uu_2^2 + 50u_1^2u_2) \\
&\quad + (20u^4u_3 + 170u^3u_1u_2 + 140u^2u_1^3) \\
&\quad + (30u^6u_2 + 140u^5u_1^2) + 25u^8u_1 + u^{11}
\end{aligned}$$

Polynomials a_n, $w(u_j) = 2j + 1$

Experiment

In contrast to the Burgers hierarchy case, here we do not know an explicit formula for a_n, but this is not too important, the main problem is to guess what are the objects which we are counting.

Let us pass to a less detailed statistics by gluing together terms of the same degree. Polynomials $a_n(u,\ldots,u)=(u\partial_u+u^2)^n(u)$ contain only odd powers and their coefficients constitute the triangle of B-analogs of Stirling numbers of the second kind (OEIS:A039755)

1							**1**
1	1						**2**
1	4	1					**6**
1	13	9	1				**24**
1	40	58	16	1			**116**
1	121	330	170	25	1		**648**
1	364	1771	1520	395	36	1	**4088**

The sums in rows, that is the total sums of coefficients $a_n(1,\ldots,1)$ give rise to the sequence (OEIS:A007405) of B-analogs of the Bell numbers, or the Dowling numbers. Great! But what is it?

B type partitions (signed set partitions, $\mathbb{Z}_2$-partitions)

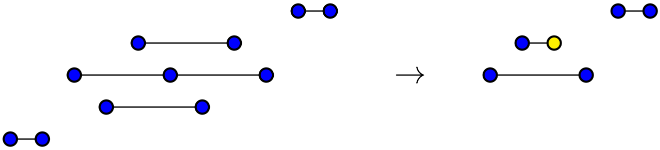
Additional structures on the set bring to special classes of set partitions. B type partitions (Dowling 1973) make use of the reflection $j \to -j$.

A partition π of the set $\{-n, \ldots, n\}$ is called the B_n type partition if:

1) $\pi = -\pi$, that is for each block $\beta \in \pi$ also $-\beta \in \pi$;
2) π contains only one block $\pi_0 \in \pi$ such that $\pi_0 = -\pi_0$.

We will denote Π_n^B the set of all such partitions and $\Pi_{n,k}^B$ those partitions which contain k block pairs.

In a brief notation, the negative elements of the 0-block are omitted, and only that block of each pair is displayed for which the element with minimal absolute value is positive; the minus signs are denoted by over bars:



$$-5, -4|-3, 0, 3|-2, 1|-1, 2|4, 5 \quad \to \quad 03|1\bar{2}|45$$

Generating operations

For a block β, let $|\beta|$ denote the number of positive elements in it:

$$|\beta| = \#\{i \in \beta : i > 0\}.$$

It is clear that the number of negative elements in the block is $|\bar{\beta}|$.

Let a partition $\pi \in \Pi_{n,k}^B$ consists of 0-block π_0 and block pairs $\pi_1, \bar{\pi}_1$, ..., $\pi_k, \bar{\pi}_k$, such that the element of π_j with minimal absolute value is positive. For such a partition, let

$$p(\pi) = u_{|\pi_0|} \cdot u_{|\pi_1|-1} u_{|\bar{\pi}_1|} \cdots u_{|\pi_k|-1} u_{|\bar{\pi}_k|}.$$

Theorem 3. The polynomials (6) are equal to

$$a_n = \sum_{\pi \in \Pi_n^B} p(\pi).$$

Thus, a_n are the $\mathbb{Z}_2$-analogs of the full exponential Bell polynomials Y_n.

Example: $n=3$	u_3	$5u^2u_2$	$8uu_1^2$	$9u^4u_1$	u^7
	0123	$0\|123$	$0\|12\bar{3}$	$0\|12\|3$	$1\|2\|3\|4$
		$0\|1\bar{2}\bar{3}$	$0\|1\bar{2}3$	$0\|1\bar{2}\|3$	
		$012\|3$	$01\|23$	$0\|13\|2$	
		$013\|2$	$01\|2\bar{3}$	$0\|1\bar{3}\|2$	
		$023\|1$	$02\|13$	$0\|23\|1$	
			$02\|1\bar{3}$	$0\|2\bar{3}\|1$	
			$03\|12$	$01\|2\|3$	
			$03\|1\bar{2}$	$02\|1\|3$	
				$03\|1\|2$	

Proof. Let us denote the sum in the right hand side p_n. Obviously, $p_0 = u = a_0$, so we only have to prove that p_n satisfy the same recurrent relations as a_n, that is, $p_n = (D + u^2)(p_{n-1})$.

Notice, that deleting of elements $\pm n$ from any B_n partition gives us a B_{n-1} type partition. Therefore, Π_n^B is constructed from Π_{n-1}^B by adding $\pm n$ in all possible ways which are described by the following operations:

$d_0 : \Pi^B_{n-1,k} \to \Pi^B_{n,k}$, insertion of both elements $\pm n$ into 0-block;

$d_j : \Pi^B_{n-1,k} \to \Pi^B_{n,k}$, $j = 1, \ldots, k$, insertion of $\pm n$ into blocks $\pm\pi_j$;

$\bar{d}_j : \Pi^B_{n-1,k} \to \Pi^B_{n,k}$, $j = 1, \ldots, k$, insertion of $\pm n$ into blocks $\mp\pi_j$;

$M : \Pi^B_{n-1,k} \to \Pi^B_{n,k+1}$, adding of the new block pair $\{-n\}$, $\{n\}$.

Starting from the trivial partition of the set $\{0\}$, these operations generate all B type set partitions, in a unique way. Let us keep track of the monomial $p(\pi)$, $\pi \in \Pi^B_{n-1,k}$, under these operations:

d_0: the factor $u_{|\pi_0|}$ is replaced with $u_{|\pi_0|+1}$;

d_j: the factor $u_{|\pi_j|-1}$ is replaced with $u_{|\pi_j|}$;

$\bar{d}_j$: the factor $u_{|\bar{\pi}_j|}$ is replaced with $u_{|\bar{\pi}_j|+1}$;

M: two new factors u are added.

Therefore, application of all possible operations maps the monomial $p(\pi)$ to the sum of monomials $(D + u^2)(p(\pi))$. ■

Korteweg–de Vries hierarchy

Recurrent relations

A most effective computation method for the KdV flows is as follows (see proof e.g. in GD 1975). The Riccati equation

$$D(f) + f^2 = \lambda - u, \quad \lambda = z^2/4 \tag{7}$$

defines the generating function

$$f(z) = -\frac{z}{2} + \frac{f_1(u)}{z} + \frac{f_2(u, u_1)}{z^2} + \cdots + \frac{f_n(u, \ldots, u_{n-1})}{z^n} + \cdots \ .$$

Let

$$g(z) = \frac{1}{2(f(z) - f(-z))} = -\frac{1}{2z} - \frac{g_1}{z^3} - \frac{g_3}{z^5} - \cdots - \frac{g_{2m-1}}{z^{2m+1}} - \cdots$$

then the KdV hierarchy is

$$u_{t_{2m+1}} = D(g_{2m+1}) = u_{2m+1} + \ldots, \quad m = 0, 1, 2, \ldots \ .$$

Equation (7) amounts to the recurrent relations

$$f_1 = u, \quad f_{n+1} = D(f_n) + \sum_{s=1}^{n-1} f_s f_{n-s}, \quad n = 1, 2, \ldots \tag{8}$$

which are the main object for us. The equation for $g(z)$ is equivalent to

$$g_1 = u, \quad g_{2m+1} = f_{2m+1} + 2\sum_{s=1}^{m} g_{2s-1} f_{2m-2s+1}, \quad m = 1, 2, \ldots$$

$$f_1 = u$$
$$f_2 = u_1$$
$$f_3 = u_2 + u^2$$
$$f_4 = u_3 + 4uu_1$$
$$f_5 = u_4 + (6uu_2 + 5u_1^2) + 2u^3$$
$$f_6 = u_5 + (8uu_3 + 18u_1u_2) + 16u^2u_1$$
$$f_7 = u_6 + (10uu_4 + 28u_1u_3 + 19u_2^2) + (30u^2u_2 + 50uu_1^2) + 5u^4$$

Polynomials f_n, $w(u_j) = j + 2$

First interpretation: unexpanded monomials

Let us consider expressions φ builded from the variable u and operations $M(a,b)$, $d_j(a)$, $1 \le j \le \deg a$ where $\deg a =$ number of instances of u in a.

The value of expression $\operatorname{expand}(\varphi)$ is computed as follows:
— independently on the order of operations, all d_j are applied before M;
— $d_j(a)$ acts by replacing of j-th instance of u_i in a with u_{i+1} (u is identified with u_0, as usual);
— $M(a,b)$ is replaced by the product ab.

Let Φ_n denote the set of all expressions with the total number of symbols u, d, M equal to n. For instance:

	unexpanded monomials	expanded monomials
$n=1$	u	u
$n=2$	$d_1(u)$	u_1
$n=3$	$d_1(d_1(u)),\quad M(u,u)$	$u_2,\quad u^2$
$n=4$	$d_1(d_1(d_1(u))),$	$u_3,$
	$d_1(M(u,u)),\quad d_2(M(u,u))$	$uu_1,\ uu_1$
	$M(d_1(u),u),\quad M(u,d_1(u))$	$uu_1,\ uu_1$

Theorem 4. The number of different expressions builded from symbols M, d_j, u with the same monomial as their value is equal to the coefficient of this monomial in polynomials f_n. In other words,

$$f_n = \sum_{\varphi \in \Phi_n} \operatorname{expand}(\varphi). \tag{9}$$

Proof. We make use of the properties

$$\sum_{j=1}^{\deg a} \operatorname{expand}(d_j(a)) = D(\operatorname{expand}(a)),$$

$$\operatorname{expand}(M(a,b)) = \operatorname{expand}(a) \operatorname{expand}(b).$$

Any expression from Φ_{n+1}, $n > 0$ is either of the form $d_j(a)$ where $a \in \Phi_n$, $1 \le j \le \deg a$ or of the from $M(a,b)$ where $a \in \Phi_s$, $b \in \Phi_{n-s}$. This implies that polynomials (9) satisfy the recurrent relation (8). ■

This interpretation is fairly intuitive, but it is desirable to compare it with something more standard.

Experiment

As usual, let us identify the terms of the same power in f_n. This yields an integer triangle which, apparently, is not in the OEIS. However, the coefficients sum totals turn out to be known: $f_{n+1}[1]$ is equal to the number of non-overlapping partitions of the set $[n]$, or the Bessel number B_n^* (OEIS:A006789)

$$\begin{array}{lllllll}
1 & & & & & & \mathbf{1} \\
1 & & & & & & \mathbf{1} \\
1 & 1 & & & & & \mathbf{2} \\
1 & 4 & & & & & \mathbf{5} \\
1 & 11 & 2 & & & & \mathbf{14} \\
1 & 26 & 16 & & & & \mathbf{43} \\
1 & 57 & 80 & 5 & & & \mathbf{143} \\
1 & 120 & 324 & 64 & & & \mathbf{509} \\
1 & 247 & 1170 & 490 & 14 & & \mathbf{1922} \\
1 & 502 & 3948 & 2944 & 256 & & \mathbf{7651} \\
1 & 1013 & 12776 & 15403 & 2730 & 42 & \mathbf{31965}
\end{array} \tag{10}$$

Notice, that equation $u\partial_u(f) + f^2 = \lambda - u$ is equivalent to the Bessel equation, indeed. Moreover, one can see in the triangle the Euler numbers (OEIS:A000295), the Catalan numbers (OEIS:A000108) and powers of 4.

Second interpretation: non-overlapping partitions

This class of set partitions (Flajolet, Schott 1990) engages the order relation on the partitioned set $[n] = \{1, \ldots, n\}$.

Blocks α and β of a set partition π overlap if

$$\min \alpha < \min \beta < \max \alpha < \max \beta.$$

The partition is non-overlapping (NOP) if any two blocks in it do not overlap. All NOPs of the set $[n]$ will be denoted Π_n^*.

The interval $[\min \alpha, \max \alpha]$ is called the support of the block α. The definition of NOP is equivalent to the property that supports of any two blocks either do not intersect or lie one in another.

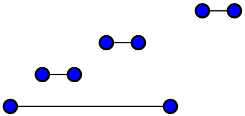
Remark. A more restrictive condition of non-crossing forbids $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$. NCPs are more popular that NOPs, but their relation with integrable equations is an open question at the moment.

Some simple properties of NOPs:

— At $n < 4$, we have $\Pi_n^* = \Pi_n$; in Π_4, only one partition $13|24$ overlaps.

— Singletons do not overlap with any block.

— NOPs with doublets only are identified with the balanced sets of parentheses:



$\rightarrow$ (( ) ( )) ( )

This explains the Catalan numbers in the above number triangle. The recursion for the 'dispersionless terms' appears if we erase the differentiation:

$$f_1 = u, \quad f_{n+1} = \cancel{D(f_n)} + \sum_{s=1}^{n-1} f_s f_{n-s} \quad \rightarrow \quad u, 0, u^2, 0, 2u^3, 0, 5u^4, 0, \ \ldots$$

But, how to establish a correspondence with generic NOPs?

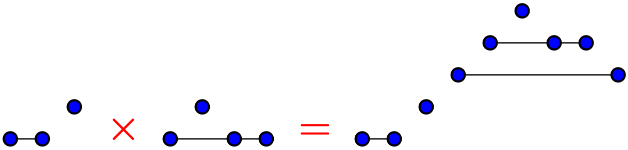
Generating operations

Let us identify u with the partition $\{\varnothing\}$ and define the action of M and d_j on partitions, in such a way that expressions Φ_{n+1} be in a one-to-one correspondence with Π^*_n.

Degree. Let $\deg \pi = k$ if π contains $k-1$ multiplets.

Operation M. Let $\rho \in \Pi^*_r$, $\sigma \in \Pi^*_s$. Denote by $(\sigma)_{r+1}$ the partition of the set $\{r+2, r+s+1\}$ obtained from σ by adding $r+1$ to each element, and define

$$M(\rho, \sigma) = \rho \cup \{\{r+1, r+s+2\}\} \cup (\sigma)_{r+1} \in \Pi^*_{r+s+2}.$$



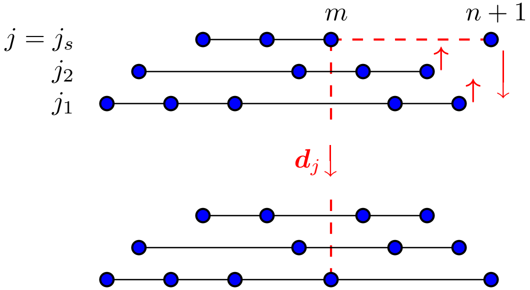
In particular, if $\rho = \{\varnothing\}$ then $(\sigma)_1$ is bounded by the doublet $\{1, s+2\}$, and if $\sigma = \{\varnothing\}$ then the doublet $\{r+1, r+2\}$ is appended to ρ.

Notice that $\deg M(\rho, \sigma) = \deg \rho \deg \sigma$.

Operation d_j. Adding of the element $n+1$ to $\pi \in \Pi_n^*$.

If $j = 1$ then the element is added as a singleton.

If $1 < j \le k = \deg \pi$ then let us denote $\mu_2, \ldots, \mu_k$ all multiplets in π, ordered by increase of their minimal elements. Assume that all blocks with support containing μ_j are enumerated by a sequence $j_1 < \cdots < j_s = j$. Operation d_j rotates parts of these blocks as shown on the diagram while all the rest blocks do not change.



More formal: divide μ_{j_r} into left and right parts with respect to $m = \max \mu_j$

$$\mu_{j_r}^{-} = \{i \in \mu_{j_r} : i < m\}, \quad \mu_{j_r}^{+} = \{i \in \mu_{j_r} : i \geq m\}$$

and form the new blocks

$$\tilde{\mu}_{j_1} = \mu_{j_1}^{-} \cup \{m, n+1\}, \quad \tilde{\mu}_{j_r} = \mu_{j_r}^{-} \cup \mu_{j_{r-1}}^{+}, \quad r = 2, \ldots, s.$$

Theorem 5. Operations M, d_j generate any NOP, in a unique way.

Proof. Given a partition, the sequence of operations which brings to it is recovered by consideration of the block β containing the maximal element of the partition. If β is a singleton then the last operation was d_1; if a doublet then it was M; if a multiplet then it was d_j where j is the maximal number such that the support of multiplet μ_j contains the last to the end element of β. In each case, applying of inverse operation brings to NOPs with lesser numbers of elements. ∎

The established bijection allows to associate a certain monomial with each NOP, although not in a quite effective way, because we first have to build an exression $\varphi \in \Phi_n$ corresponding to $\pi \in \Pi^*_{n-1}$ and then to compute $\mathrm{expand}(\varphi)$:

$$\begin{array}{rccc} & \Phi_n & \leftrightarrow & \Pi^*_{n-1} \\ \mathrm{expand} & \downarrow & \swarrow & \\ & f_n & & \end{array}$$

Nevertheless, it is easy to trace at the degree of monomial which is one more than the number of multiplets in the partition:

Corollary. The number of NOPs of n elements containing k multiplets is equal to the number in the n-th row and k-th column of the number triangle (10), starting their enumeration from 0.

Conclusion

- Too few examples to make far-reaching conclusions at the moment.
- A conjecture is that any polynomial (?) integrable hierarchy corresponds to some type of combinatorial objects, possibly unknown. Then, what is the combinatorics related to the mKdV, 5-th order KdV-likes, nonlinear Schrödinger equation and so on?
- In contrast, the objects in the combinatorics are so plentiful and diverse that it seems doubtful that any one can be associated with an integrable hierarchy. This property should be very special.
- It is important to understand what is integrability intermediately in combinatorial terms (rather than of the level of generating functions). For instance, is it possible to get a proof of the commutativity of the flows based on their combinatorial interpretation?

My procedure was this: I would count the stones by eye and write down the figure. Then I would divide them into two handfuls that I would scatter separately on the table. I would count the two totals, note them down, and repeat the operation.

Borges, Blue tigers (translated by Andrew Hurley)