The `psum` package

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Summary

The generalized summation by parts algorithm provided in this *Mathematica 5.0* package is intended for solving of difference equations

$$T^m(y) - ay = b. \tag{1}$$

Here, the coefficients a, b and unknown function y depend on a finite number of variables u_j, and T is the shift operator

$$T : f(u_i, \dots, u_j) \to f(u_{i+1}, \dots, u_{j+1}).$$

Possible applications are related with testing of integrability of differential-difference equations

$$\partial_t(u_n) = f(u_{n-m}, \dots, u_{n+m}), \quad n \in \mathbb{Z}. \tag{2}$$

Descriptions of the algorithm and the integrability test can be found in:

> V.E. Adler. Integrability test for evolutionary lattice equations of higher order. arXiv:1408.5726, 25 Aug 2014;

see also the references within and my talk with the same title (in Russian).

Filelist

`psum.nb, .m`	implementation of the algorithm;
`examples.nb`	basic usage; time tests; comparison with the discrete homotopy operator (in the case of equation $T(y) - y = b$);
`tests.nb`	integrability tests for several lattice equations (2);
`Yamilov.nb`	full list of integrable equations (2) at $m = 1$, according to Yamilov, J. Phys. A 39 (2006) R541.

Usage

Copy the files into your working directory. In the Mathematica session, the package is loaded by the commands like

```
SetDirectory["c:\\users\\bilbo\\programs"]
<<psum`
```

This makes available the following functions:

- `psum[m,a,b]` returns `{y, obstacle}`;
  - if `obstacle` $\neq 0$ then a solution of (1) does not exist;
  - if `obstacle` $= 0$ then `y` is the general solution of (1);
  - `const` denotes an arbitrary constant contained in the general solution if $a = T^m(h)/h$; otherwise, the solution is unique;
- auxiliary functions `T[f, k]`, `vars[f]`, `ords[f]` are used in the definition of `psum` and also in `tests.nb`.

Limitations

The correctness of the computations relies on the definitions of the functions `cf[f]` (canonical form) and `X[f,k]` (extraction) which are applied to expressions somewhere in the guts of the algorithm. These functions are not protected. The default definitions

```
cf[f_] := Together[f]
X[f_, k_] := Integrate[cf[D[f, u[k]]], u[k]]
```

are suitable if the coefficients a, b are rational functions. In more complicated situations (for expressions containing functions with non-trivial identities), this may require redefining. In general, if `f` does not actually depend on some of `u[j]` then `cf[f]` should cancel out all such variables.

Compatibility

The package works for Mathematica 5 and 8. It has not been tested for other versions.